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PARAMETRIC AND NONLINEAR MODE INTERACTION BEHAVIOUR IN THE DYNAMICS OF STRUCTURES.

by

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The report is concerned with the resonant behaviour of general structural systems under single frequency harmonic excitation. In particular it is concerned with the parametric and nonlinear phenomena which arise when there are certain integral relationships between the excitation frequency and some of the natural frequencies of the structure (external resonance), and amongst the natural frequencies themselves (internal resonance).

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In the theory the equations of motion retaining linear parametric terms and including quadratic inertial nonlinearities are considered. A general theoretical approach is presented which is applicable to any multimode structure. The theory predicts the resonance conditions, and in the vicinity of each yields simpler equations giving an approximate solution. Generally these equations require a numerical solution for a given structure. In certain cases further analysis on a reduced form of the equations yields analytical expressions which define the boundaries of the resonance region in the excitation amplitude-frequency plane.

The theory is applied to lumped mass mathematical models (based on experimental models) and is verified in the simpler cases at least by the direct numerical integration of the original equations of motion.

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The report is concerned with the resonant behaviour of general structural systems under single frequency harmonic excitation. In particular it is concerned with the parametric and nonlinear phenomena which arise when there are certain integral relationships between the excitation frequency and some of the natural frequencies of the structure (external resonance), and amongst the natural frequencies themselves (internal resonance).

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CONTENTS

PRINCIPAL NOTATION

a	Excitation amplitude (displacement)
$A_r(t)$	Slowly varying modal amplitude
$A_{ro}(t)$	D.C. (zero frequency) slowly varying amplitude
$[A]$	Inertia matrix
$[C]$	Stiffness matrix
f_r	$= F_r/(\omega_r^2 - \Omega_f^2)$, linear forced response component
F_r	Excitation amplitude (force, normal coordinates)
FF	Excitation frequency (computer diagrams)
F1,2	Response frequencies (computer diagrams)
FN1,2	Natural frequencies (computer diagrams)
ℓ_r	Beam length
ℓ_{ijk}	Nonlinear coefficients (quadratic)
m_r	Mass
m_{ijk}	Nonlinear coefficients (quadratic)
N_r	$= f(\ell_{rij}, m_{rij}, \Omega_j)$. Autoparametric terms in variational equations
p_r	Normal coordinate
P_r	$= f(\kappa_{rj}, \ell_{rjk}, m_{rjk}, \Omega_k)$ parametric type terms in the variational equations
$[R]$	Modal matrix
SC1,2	Amplitude scaling factors (computer diagrams)
t	Time
α	Negative damping coefficient
Δ_r	$= (\Omega_r^2 + \Delta_r - \omega_r^2)$, detuning parameter
∇_r	$= \sum_{i=1}^{n} \epsilon \ell_{rir} A_{io} \Omega_r^2$, D.C. additions to variational equations
ϵ	Small parameter
κ_{rj}	Parametric term

λ_r	Beam stiffness
$\phi_r(t)$	Slowly varying modal phase
ψ_r	Forced response phase
ω_r	Natural (linear) frequency
Ω_r	Response frequency
Ω_f	Excitation frequency

1. INTRODUCTION

1.1 GENERAL

This report is concerned with some aspects of the 'resonant' response of structures in situations in which parametric and nonlinear influences, which are always present, become of considerable significance. In such situations response estimations using the usual constant coefficient linear modelling of the structure are quite useless. The 'smallness' of the nonlinear terms or the time varying coefficients is not sufficient to prevent them from having an overwhelming effect on the response in the course of time. In many engineering situations the excitation is applied to the structure in a more or less steady oscillatory fashion for an appreciable duration thus allowing the development of the influence of these effects.

In more precise mathematical modelling of structures in dynamic conditions, nonlinear quadratic and higher order inertial terms are nearly always present. The adjective 'inertial' is applied to these terms which have their origin in the kinetic energy T of the structure. The existence of quadratic nonlinearities from this source is very important but curiously it has received little attention in the literature. Elastic nonlinearities have received much more consideration but they are usually a symmetric form so that they appear as odd functions through cubic and quintic nonlinear terms.

This report concentrates on behaviour characteristics which arise from or can be explained by the existence of the quadratic inertial nonlinearities. The higher orders of nonlinearity produce their own phenomena and as, in certain excitation frequency bands, the linear

model has to be augmented by consideration of quadratic nonlinearities so this model in turn may also require cubic nonlinearities to be included to produce qualitatively certain phenomena or quantitatively a required degree of accuracy. Fortunately it seems to be the case that in general the influence of higher and higher orders of nonlinearity of, presumably, smaller and smaller magnitude becomes virtually undetectable in the presence of damping so that an examination of terms of higher order than the cubics is seldom necessary to explain observed behaviour.

In structural dynamic analysis 'ordinary' forced excitation, an independent time function constituting a nonhomogeneous 'right hand side' to the equations of motion, is usually the only forcing action considered. However more refined analysis indicates in virtually any situation the existence of parametric forcing terms which involve both the force and the resulting motion together and constitute homogeneous terms with time dependent coefficients. In fact even finer analysis will show that these terms in turn owe their origins to nonlinear coupling terms between two types of motion where one of the types involved has been approximated by a chosen time function. Thus the classical problem of parametric excitation (Belaiev's problem) of a beam under periodic axial excitation is, at a more precise level, a question of the nonlinear interaction of axial and bending motions: at a lower level, where the axial displacement is taken as a known function of time resulting from the given excitation, it becomes a problem of bending motion with parametric excitation.

Parametric excitation terms behave like quadratic nonlinearities in many ways, this feature is brought out in the report. In the heirarchy of 'small terms' quadratic nonlinearities and parametric terms have about the same placing although of course the parametric

terms are basically linear. If the constant coefficient structural model with ordinary forcing is regarded as the most primitive form then the next stage of refinement is that model with quadratic nonlinearities and parametric forcing included. It is about such a level of modelling and the phenomena which it introduces under certain types of excitation that this report is concerned.

Excitation itself can be subdivided into two main categories, stochastic and deterministic. These in turn can be reduced, stochastic to, among others, stationary wide band, stationary narrow band, and non-stationary, while deterministic excitation can be monofrequency sinusoidal, multifrequency periodic (square wave for instance), multi-frequency quasi-periodic (such as $\sum_n f_n \cos\Omega_n t$) and non-periodic or transient.

If attention is restricted to deterministic excitation then both the ordinary and the parametric excitation can exist in any of the above categories. There is clearly a wide range of possible forced vibration problems and very little is known about any of them. The present investigation considers both the ordinary and parametric excitation to be of the simplest type, monofrequency sinusoidal, and also considers them to have the same frequency. The range of phenomena arising if they have different frequencies or if one or other or both of them is a multifrequency input is very greatly extended.

In the analysis which follows, the linear normal modes of the structure have been used as coordinates. For small nonlinearities this seems a reasonable choice, for highly nonlinear problems some sort of nonlinear normal mode might be more appropriate.

The report discusses two, three and four mode models but it should be understood that the analysis is not restricted to this number of degrees of freedom in the whole structure. These numbers

represent the number of interacting modes in the phenomena being described. Thus the equations relating to the "two-mode model" would apply to say the 10th and 17th modes of a complex structure if these modes had natural frequencies in or near the necessary 2:1 ratio.

1.2 SPECIFIC

As mentioned in the previous section, this investigation is concerned with the resonant behaviour of general multimode structural systems, inclusive of quadratic inertial nonlinearities and linear parametric terms, subjected to monofrequency sinusoidal excitation. A formal derivation of the relevant general equations of motion is given by Barr & Nelson [1] who consider purely autoparametric interactions in structures undergoing harmonic excitation near a natural frequency, and this paper forms the basis for the current work. However, the approach presented here is more general than that given in [1] as parametric terms are included in the analysis and consideration is given to other possible external[1] resonance phenomena that arise quite apart from the direct forcing of one mode.

Qualitative experimental evidence illustrating the resonance behaviour exhibited by simple structures having appropriate internal[2] resonance conditions is presented in Section 2. The experimental approach is considered valuable in that it demonstrates the physical existence of the phenomena and provides guidelines upon which an appropriate theory can be formed. Experimental evidence showing simultaneous resonant behaviour in up to four modes is given.

[1] *Some integer relationship between the excitation frequency and the natural frequencies.*

[2] *Some integer relationship between the natural frequencies themselves.*

The general theoretical approach outlined in Section 3 is the result of development through application to idealised mathematical models, which are based on the experimental structures. The analysis of general two and three mode structural systems gives sufficient background for the theory to the expanded to any multimode system having quadratic inertial nonlinearities and parametric terms. Certain modifications to the basic 'Struble' [2] solution technique have been found to be necessary in order to maintain reasonable accuracy in the solution and to avoid problems that arise when both external and internal detuning is considered. The form of the generating solution in the Struble analysis is modified by selecting an unknown response frequency for the fundamental term in the solution, rather than the linear natural frequency, cf [1], although this unknown frequency is taken to be close to the natural frequency. In certain cases a D.C. term (zero frequency) and a forced term at the excitation frequency are also included in the generating solution. In the first approximation used the technique is essentially a limited harmonic balance with selection and retention of the more important nonlinear terms.

The results of the analysis are the 'variational' equations which are nonlinear first order differential equations describing the slowly varying amplitudes and phases of the solution. For each resonance situation in a given structure there is a corresponding set of variational equations, and all the possible sets for two and three mode structures are presented. From these the equations for any multimode system can be deduced. Generally the variational equations can only be solved numerically for a given structural system and examples of the application of the theory to structures having two

and four basic modes are given. The results are compared with the direct integration of the full equations of motion.

Further analysis of a simplified form of the variational equations in the 'hard' forcing cases, i.e. when no mode is being directly excited, yields regions in the excitation amplitude-frequency plane within which the system will exhibit resonant behaviour. Recourse has to be made to the integration of the complete variational equations in order to study the actual behaviour within the regions predicted.

2. EXPERIMENTAL WORK

2.1 EXPERIMENTAL EQUIPMENT

Figure 2.1.1 is a schematic layout of the experimental equipment. The model structure under investigation is clamped to the armature of a medium-sized electrodynamic shaker (1500 lb maximum vector thrust). The shaker is driven by an accurate sinusoidal signal generator operating through a power amplifier. Excitation intensity is monitored by an accelerometer mounted on the moving armature. The servo-control sweep oscillator can be used in place of the signal generator to perform programmed frequency sweeps with predetermined excitation levels, but it is also useful in that its inbuilt signal conditioning circuits provide a display of the actual excitation level, giving the peak level of acceleration, velocity or displacement.

The model structures are small in comparison with the size of the shaker. This is to minimise the effect of any feedback from a resonating structure causing distortion of the excitation waveform.

The response of the models themselves is monitored by strategically placed accelerometers and strain gauges. The signals from these sensors are conditioned through standard bridge and preamplifier circuits to give voltage signals that can be directly related to the behaviour of the structure. These resulting signals are manipulated by simple analog summing and scaling devices in such a way as to give signals representative of the 'modes' of the system. In free vibration each of these signals has just one frequency component, that is the natural frequency of the mode concerned. In forced vibration the signals can have two frequency components, one near the modal frequency and the other at the excitation frequency. Where required the active filter is used to remove the forced response component.

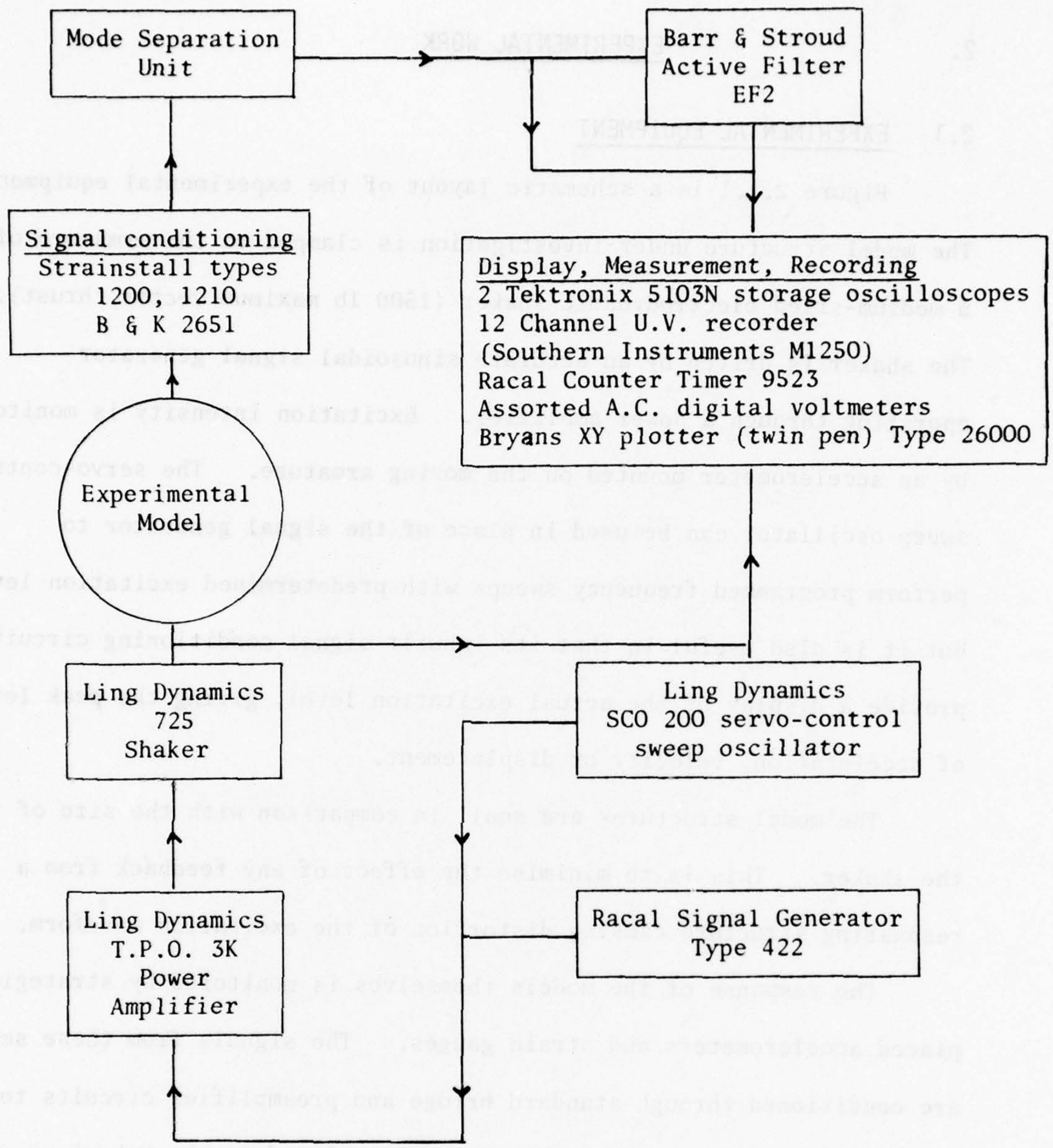


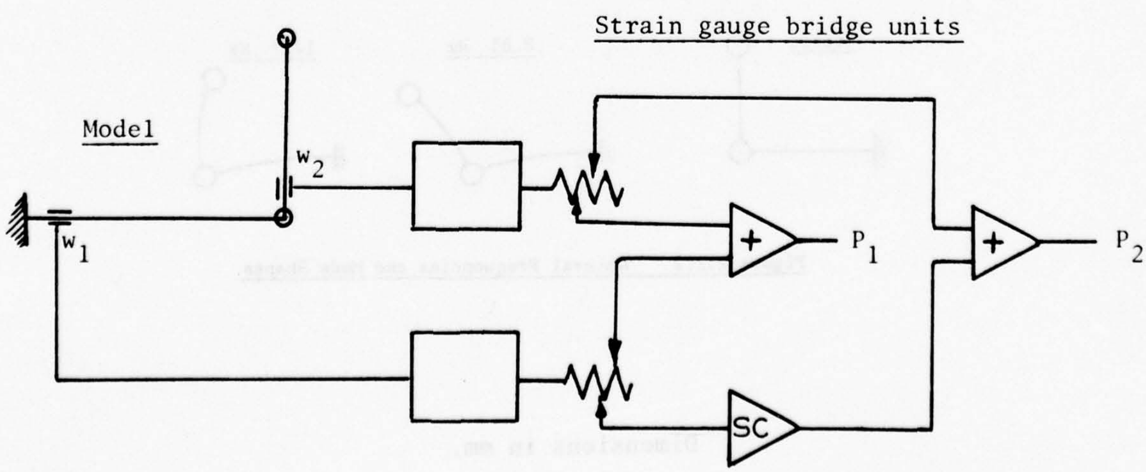
Fig. 2.1.1 EXPERIMENTAL EQUIPMENT [SCHEMATIC].

The behaviour of the modes of the system are then displayed on the oscilloscope and/or recorded on the ultraviolet recorder. A.C. voltmeters are used in the production of 'steady state' response curves. Response frequency measurement is achieved by the production of a stationary Lissajou's figure on an oscilloscope (involving the use of a further accurate signal generator).

2.2 TWO MODE MODEL

Figure 2.2.1 shows a photograph and drawing of the two mode model. The construction is such that the effective lengths of both steel strip beams are adjustable. By careful adjustment of the masses and the lengths, the two lowest natural frequencies are set in a 2-1 ratio. Figure 2.2.2 shows the natural frequencies and the type of mode shapes obtained. Strain gauge pairs sited at the roots of each beam provide signals which over a wide range are found to be linearly related to the end displacement of the beam under static end loading. These signals are directly related to the generalised coordinates to be used in the theoretical analysis of the structure.

The separation of the modes from these two signals is achieved by their manipulation in a purpose-built analog unit. Figure 2.2.3 shows the schematic layout.



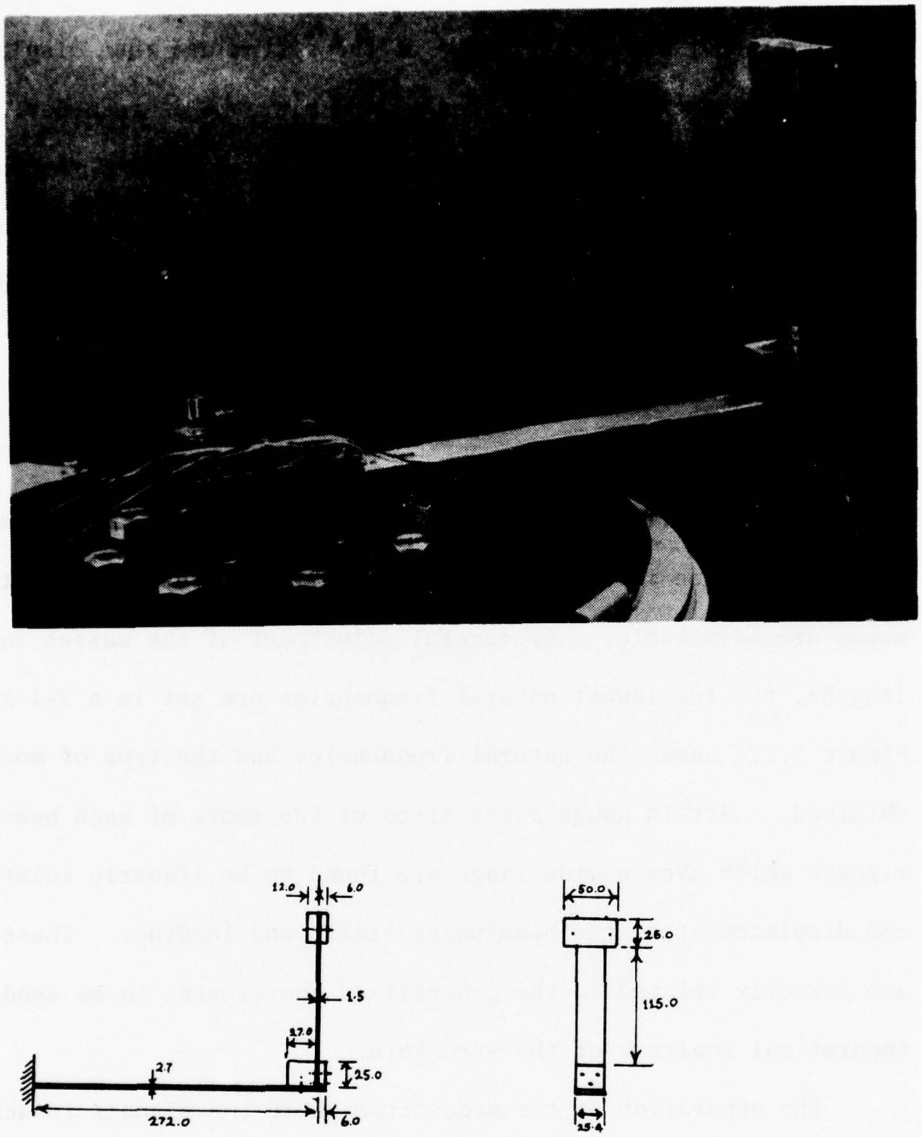


Figure 2.2.1 Two Mode Experimental Model

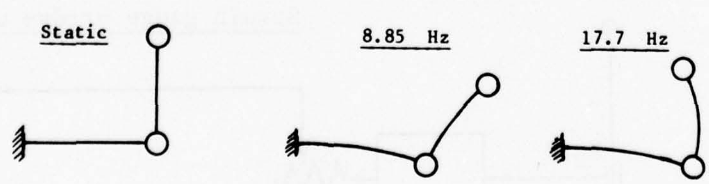


Figure 2.2.2 Natural Frequencies and Mode Shapes

Dimensions in mm.

In effect the analog devices perform the same transformation to normal coordinates P_1 and P_2 from the generalised coordinates w_1 and w_2 as does the modal matrix in the corresponding linear theory.

Figure 2.2.4 is a copy of an ultraviolet trace showing the behaviour of the model when the excitation frequency is close to the lower natural frequency. Starting from rest, growth in the lower mode is rapidly followed by growth of the higher mode. Figure 2.2.5 shows the steady state response curves obtained in the neighbourhood of this external resonance condition whilst maintaining the excitation level at 0.1 mm peak and manually sweeping up and down the frequency range from 8.5 to 9.1 Hz. The ordinary linear response curve of mode 1 is replaced by the twin-horned response curve, accompanied by a similar curve for mode 2.

Figure 2.2.6 is of the ultraviolet record taken when the excitation frequency is in the neighbourhood of the higher natural frequency. Growth in the higher mode leads to growth in the lower mode through the internal resonance condition. Figure 2.2.7 shows the steady state response curves obtained in the neighbourhood of this resonance. In this case discontinuities occur in the stationary response curves. For increasing frequency mode 2 follows the linear resonance curve to point a. At this point mode 1 jumps from zero to a high amplitude. Mode 2 then follows the curve abc and at c jumps back to the declining linear resonance curve. For decreasing frequency mode 2 follows the curve dbae, having discontinuities at b and e where mode 1 jumps in and collapses respectively. It should be noted that only discontinuities in slope occur in mode 2 at points a and b. This 'overlapping' of the response curves for increasing and decreasing frequencies is similar to that illustrated by Haxton [3] in connection with the Autoparametric Vibration Absorber. The slight

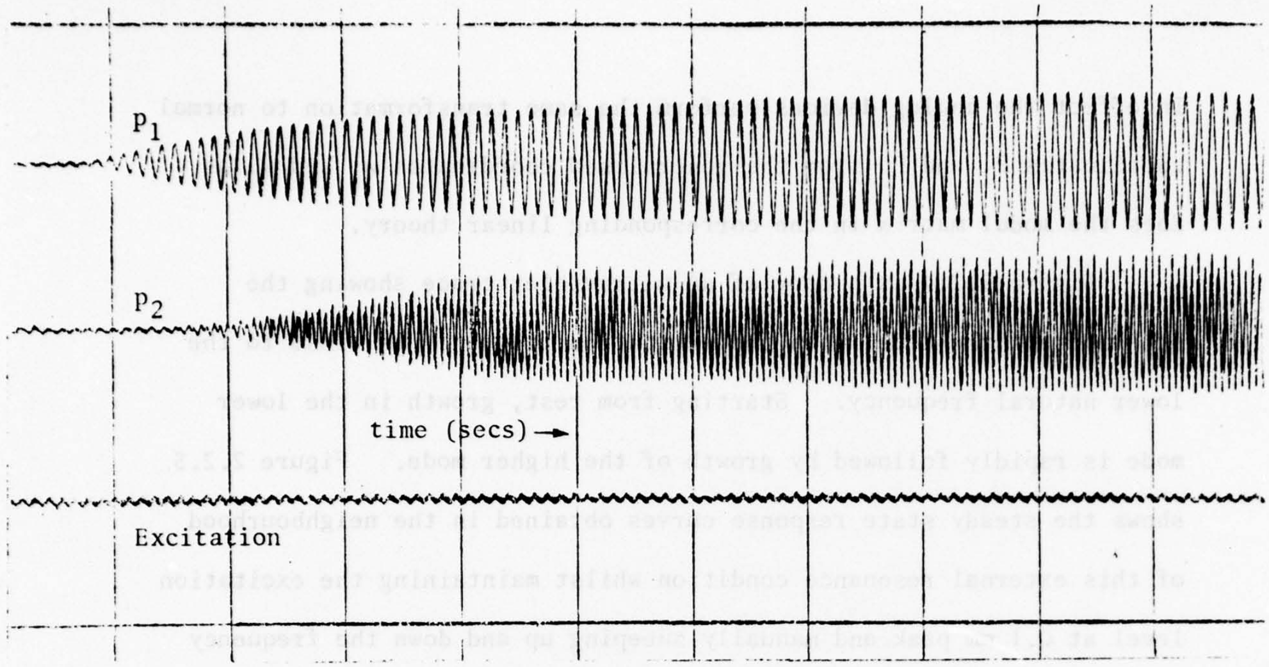


Fig. 2.2.4. Excitation of lower mode

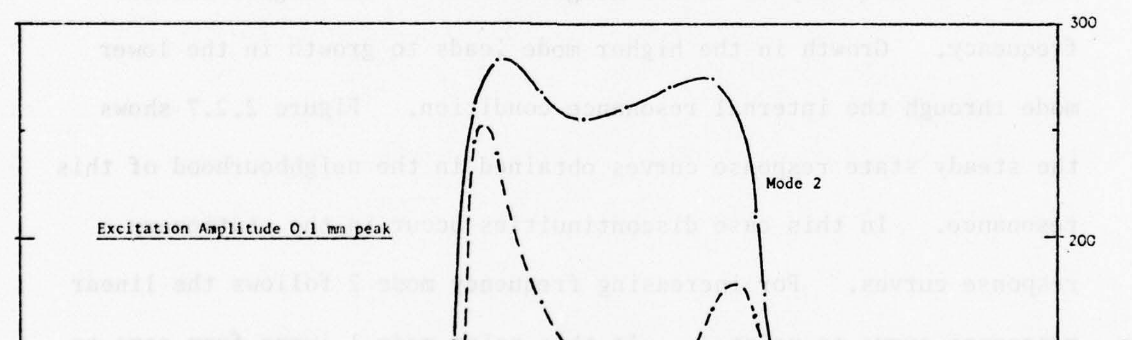


Fig. 2.2.5. Stationary resonance curve

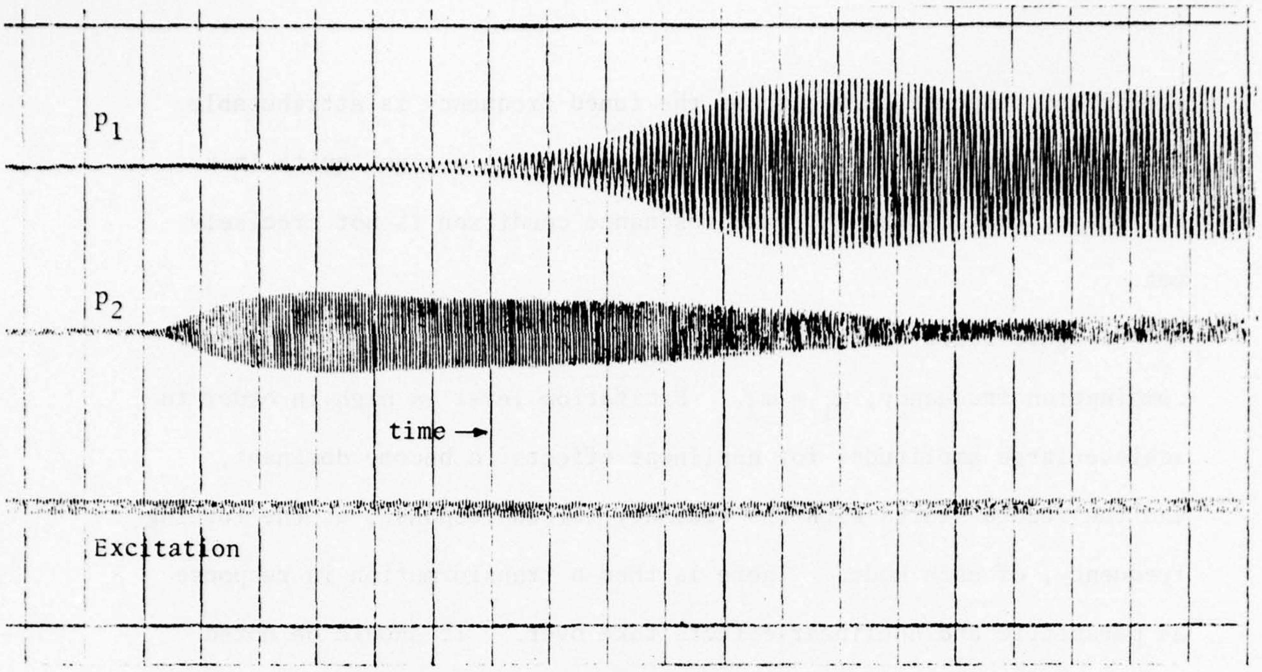


Fig. 2.2.6. Excitation of Higher Mode

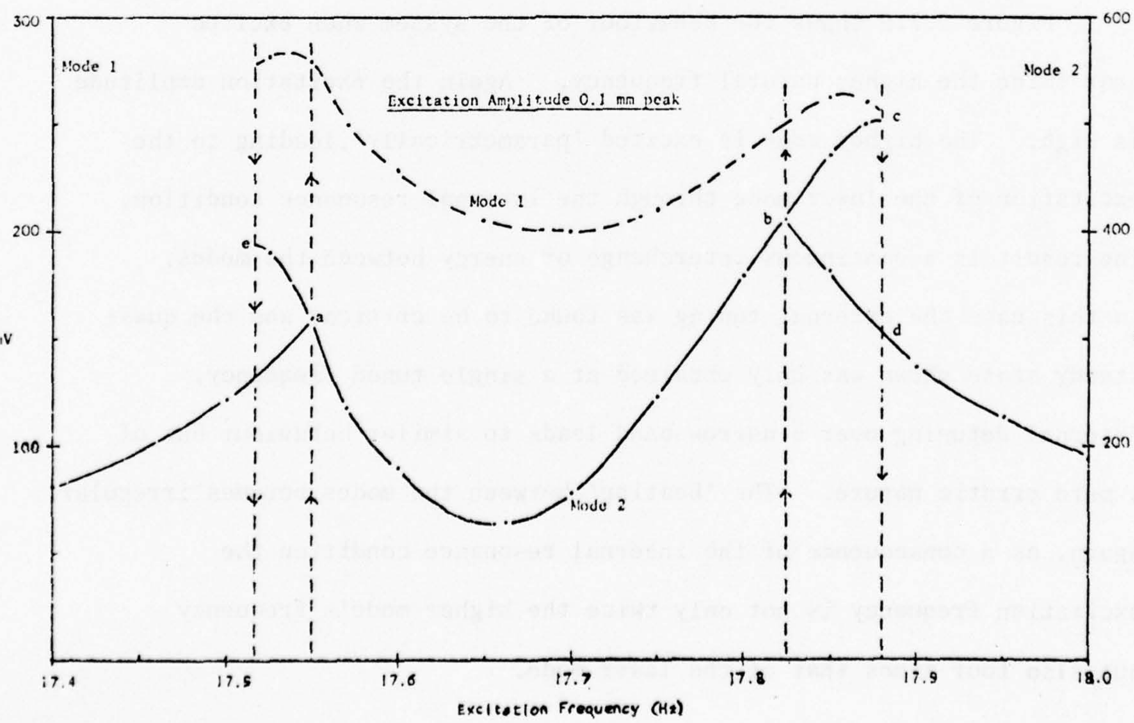


Fig. 2.2.7. Stationary resonance curves.

skewing of response curves about the tuned frequency is attributable to the two modes having natural frequencies that are not quite in a 2:1 ratio, that is the internal resonance condition is not precisely met.

Figure 2.2.8 shows the time history when excitation is at the combination frequency, $\omega_1 + \omega_2$. Excitation level is high in order to achieve large amplitudes for nonlinear effects to become dominant, and the record starts with the ordinary forced response, at the forcing frequency, of each mode. There is then a transformation in response as parametric and nonlinear effects take over. It should be noted that in this case, by virtue of the internal resonance condition, the forcing frequency is not only near the sum of the two natural frequencies, but is also three times the lower natural frequency. Also for this case moderate internal detuning of the system was found to result in desynchronisation resulting in a continuous beating between the modes.

Figure 2.2.9 shows the behaviour of the system when excited near twice the higher natural frequency. Again the excitation amplitude is high. The higher mode is excited 'parametrically',leading to the excitation of the lower mode through the internal resonance condition. The result is a continuous interchange of energy between the modes. In this case the external tuning was found to be critical and the quasi steady state shown was only obtained at a single tuned frequency. External detuning over a narrow band leads to similar behaviour but of a more erratic nature. The 'beating' between the modes becomes irregular. Again, as a consequence of the internal resonance condition the excitation frequency is not only twice the higher mode's frequency but also four times that of the lower mode.

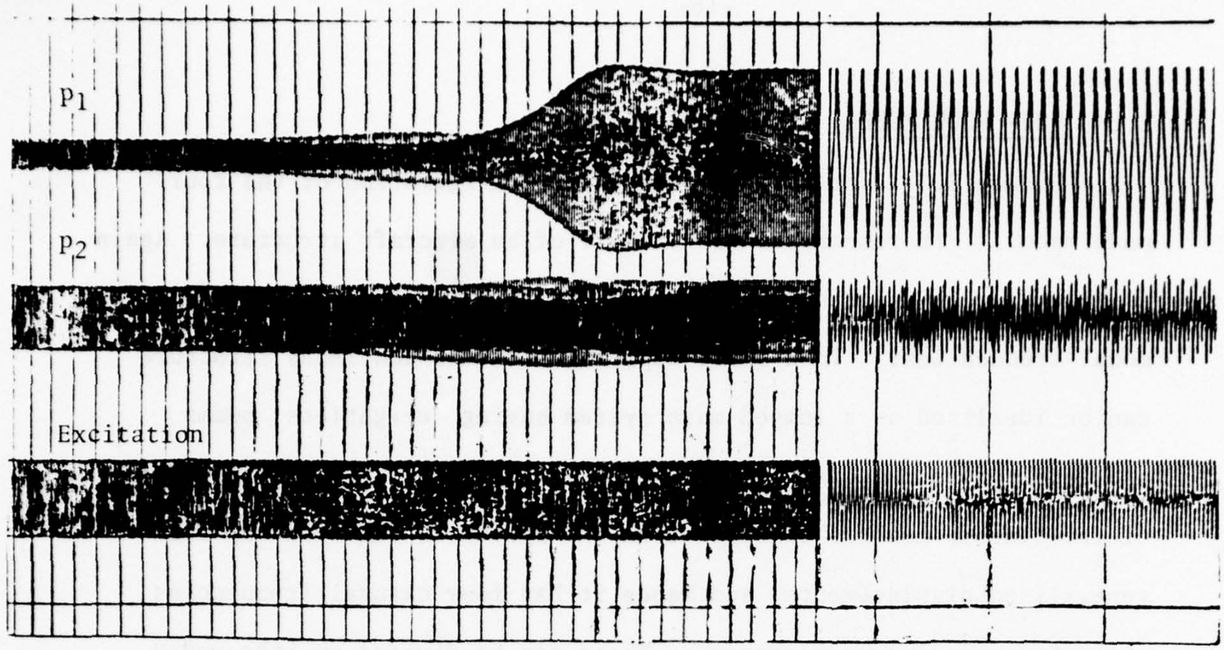


Fig. 2.2.8. Excitation at the combination frequency, $\omega_1 + \omega_2$

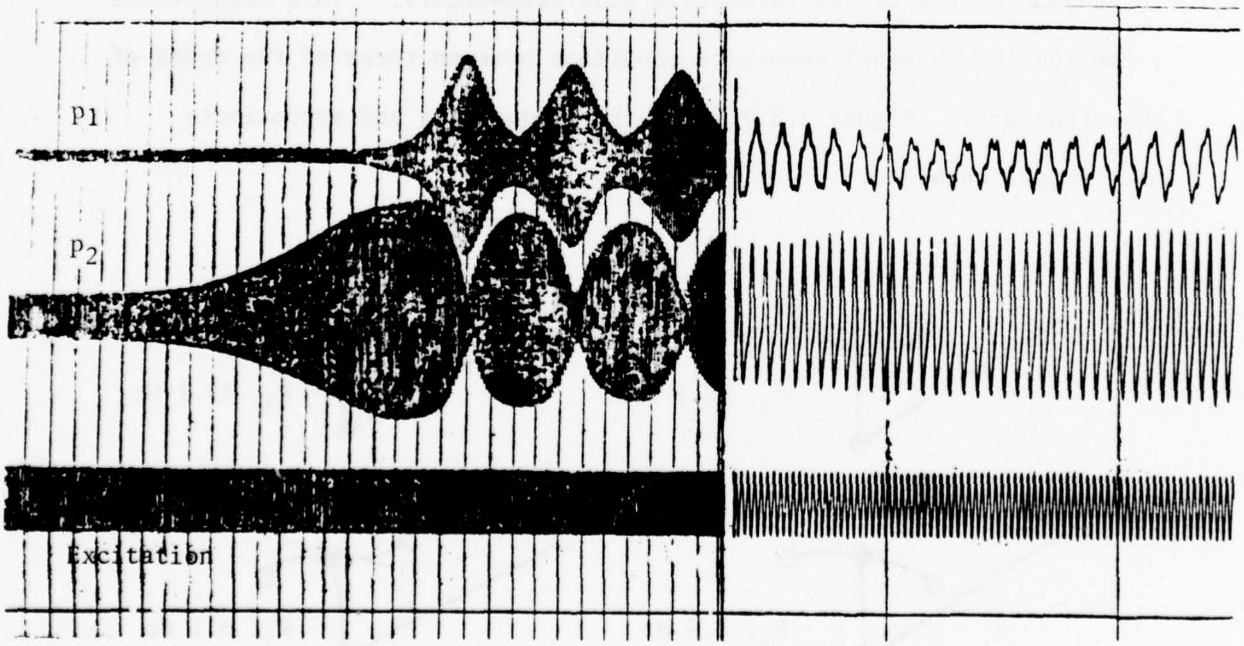


Fig. 2.2.9. Excitation at $2\omega_2$

2.3 FOUR MODE MODEL

Figure 2.3.1 shows the experimental configuration of the four mode model. It resembles the 'T' tail of an aircraft structure. Again as all the beam lengths are adjustable it is possible to set up different modal frequencies. In a first approximation the continous structure can be idealised as a lumped mass system having 'weightless' beams and four concentrated end masses. By considering only ordinary bending of the elastic members to occur then the system is described by four generalised displacements, and hence it has four natural frequencies with corresponding mode shapes. These can be divided up into modes involving symmetric and antisymmetric 'tailplane' motions, as shown in Figure 2.3.2.

Having set up an arbitrary 'fin/tailplane' configuration and measured the antisymmetric frequencies, the fuselage length is adjusted to make the highest symmetric mode frequency as near as possible equal to the sum of the two antisymmetric mode frequencies. This constitutes a combination internal resonance condition between three of the modes of the structure. Figure 2.3.2 shows the frequencies and approximate mode shapes obtained.

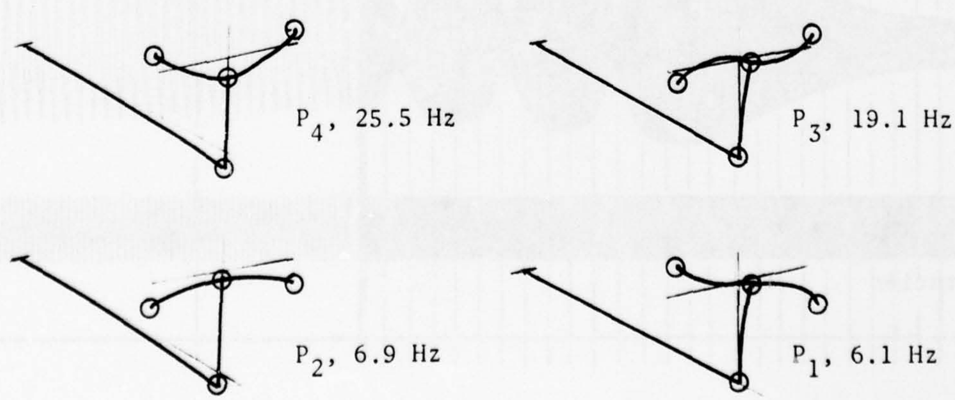


Fig. 2.3.2 EXPERIMENTAL MODE SHAPES (SCHEMATIC)

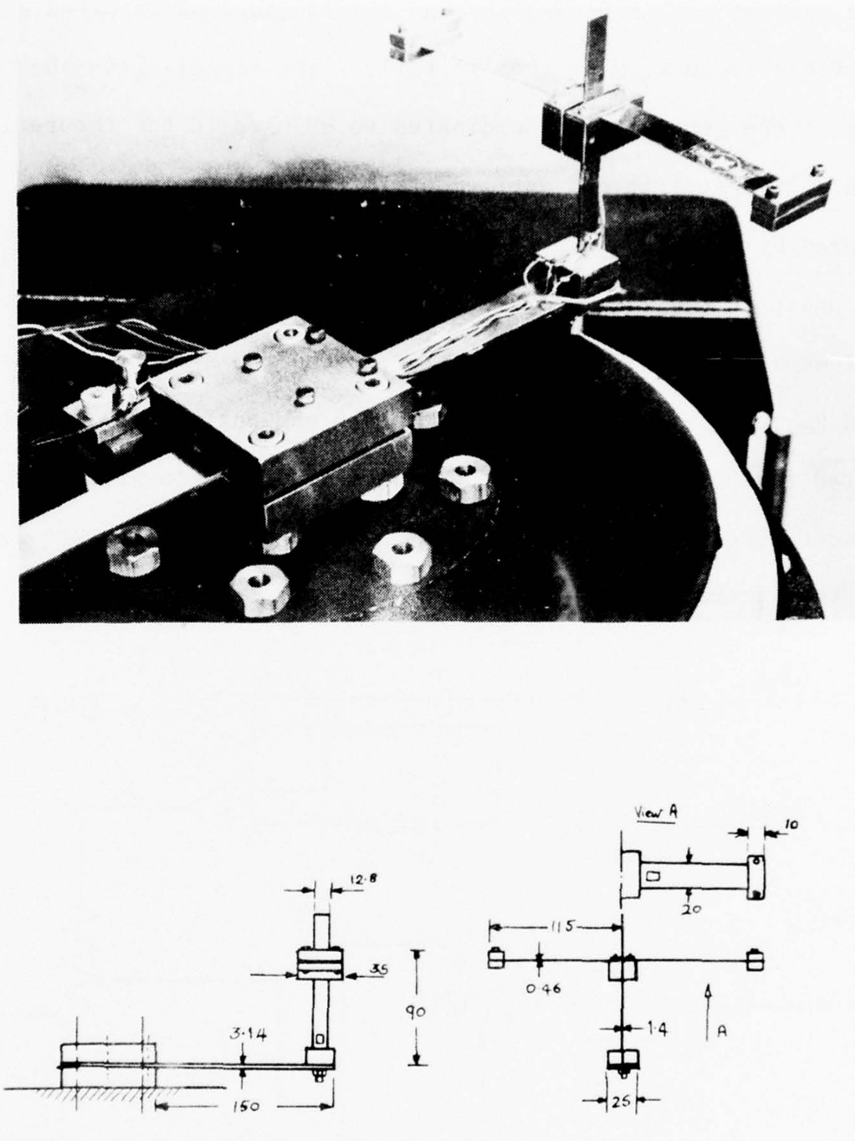


Fig. 2.3.1. The Four Mode Experimental Model

Dimensions in mm.

The behaviour of the model is monitored through the accelerometer at the base of the 'fin' and through strain gauge pairs sited at the root of each beam forming the 'T' tail. The signals from these represent the generalised coordinates to be used in the theoretical analysis. Symmetric and antisymmetric tailplane motions are monitored by summing and differencing the two tailplane signals. It was found that the accelerometer signal and the summed tailplane signal were reasonably representative of the two symmetric modes, P_4 and P_2 respectively. However the fin and antisymmetric tail signals required further manipulation to uncouple them and to give signals representative of the antisymmetric modes P_1 and P_3. Figure 2.3.3 shows the schematic layout of the equipment.

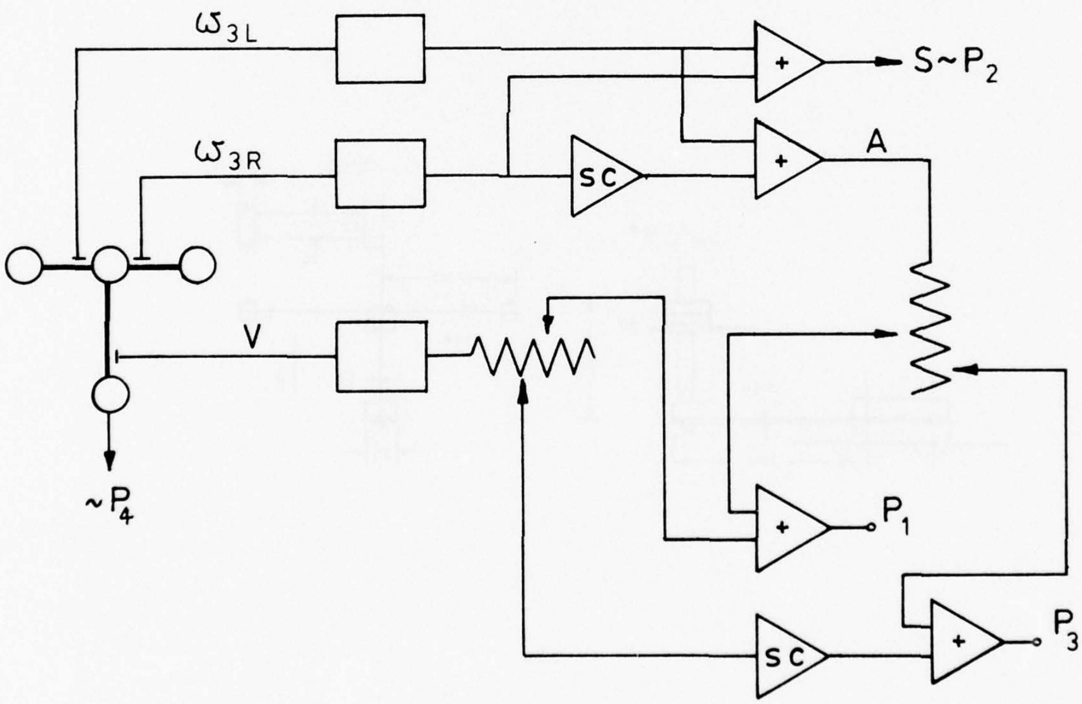
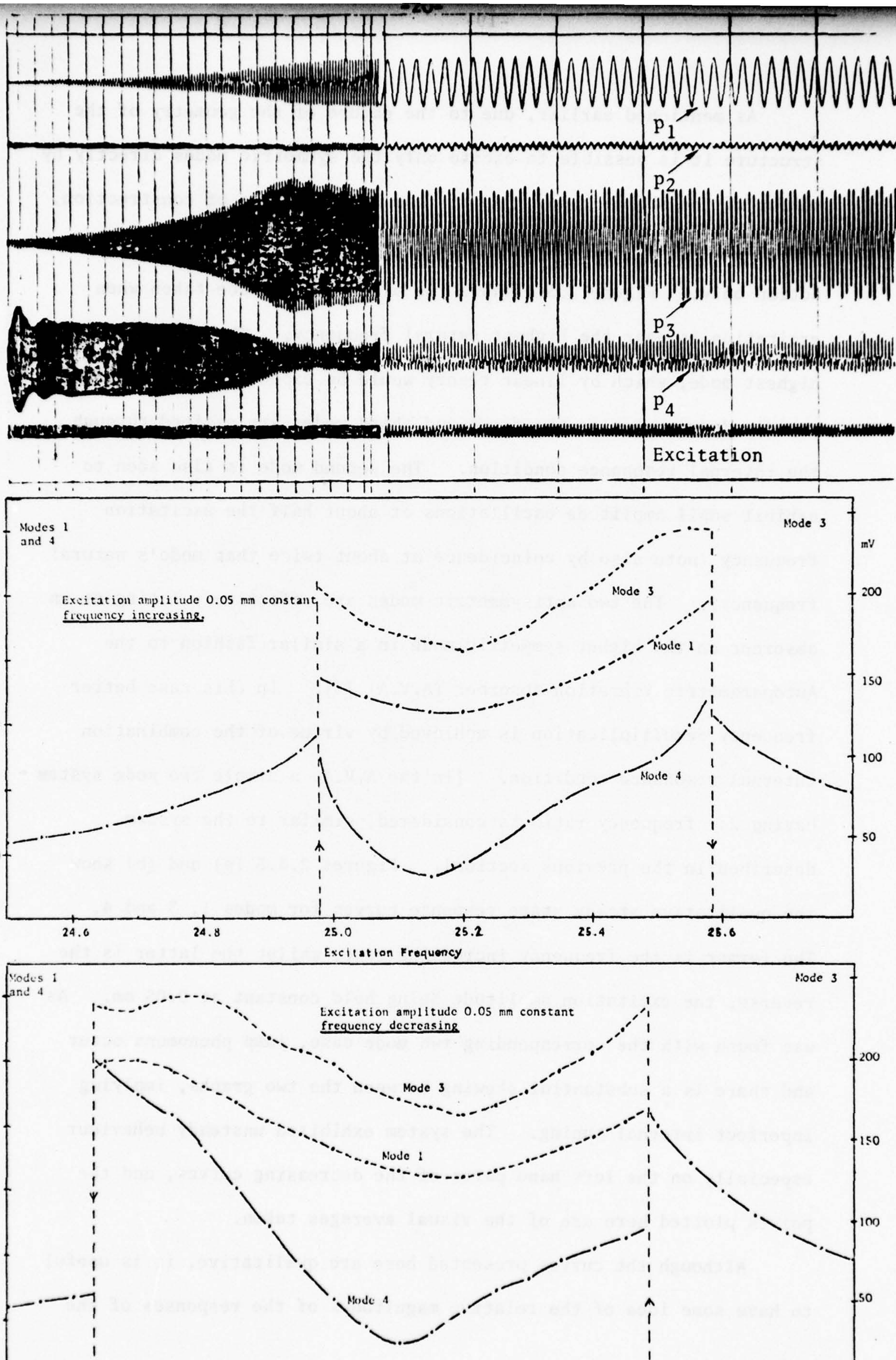


Fig. 2.3.3. Mode separation - Schematic

The variable resistors are adjusted to that in free vibration the signals P_1 and P_3 contain the single frequencies ω_1 and ω_3 only.

As mentioned earlier, due to the nature of the geometry of the structure it is possible to excite only the symmetric modes directly by motion of the shaker head. Apart from imperfections of construction, the antisymmetric modes can be excited by parametric or autoparametric action only. Figure 2.3.4 shows the ultraviolet trace taken when excitation is near the highest natural frequency. The growth of the highest mode, which by linear theory would be expected to predominate, rapidly degenerates as the first and third modes are excited through the internal resonance condition. The second mode is also seen to exhibit small amplitude oscillations at about half the excitation frequency (note also by coincidence at about twice that mode's natural frequency). The two antisymmetric modes are effectively acting as an absorber on the higher symmetric mode in a similar fashion to the Autoparametric Vibration Absorber (A.V.A) [3]. In this case better frequency demultiplication is achieved by virtue of the combination internal resonance condition. [In the A.V.A. a simple two mode system having 2:1 frequency ratio is considered, similar to the system described in the previous section]. Figures 2.3.5 (a) and (b) show the qualitative steady state response curves for modes 1, 3 and 4. The former is the frequency increasing case, whilst the latter is the reverse, the excitation amplitude being held constant at 0.05 mm. As was found with the corresponding two mode case, jump phenomena occur and there is a substantial skewing between the two graphs, implying imperfect internal tuning. The system exhibited unsteady behaviour especially on the left hand parts of the decreasing curves, and the points plotted here are of the visual averages taken.

Although the curves presented here are qualitative, it is useful to have some idea of the relative magnitudes of the responses of the



Figs. 2.3.4, 2.3.5, 2.3.6. Excitation of Mode 4

modes involved. By relating the response levels of the modes (mV) to an equivalent actual response amplitude of some part of the structure, some idea of the relative magnitudes of the responses can be achieved. The antisymmetric modes (1 and 3) can be related to peak amplitude of the fin bending displacement (at the mass), whereas the symmetric modes can be related to the peak amplitude of the 'fuselage' displacement (at the accelerometer). Hence, as an approximate calibration:

6 mm total fin bending excursion corresponds to 85 mV in mode 1
and 100 mV in mode 3

1.5 mm total fuselage bending excursion corresponds to 30 mV in Mode 4.

Surprisingly perhaps it was found that excitation of high intensity at twice the first or third modes' natural frequencies did not produce any substantial 'parametric' resonant behaviour in either of these modes. The only other resonance behaviour exhibited by the model apart from the ordinary linear forced resonance of the lower symmetric mode is when excitation is of high intensity at the summed frequency of the two symmetric modes. This constitutes a combination external resonance condition. Figure 2.3.6 is a photograph of a typical ultraviolet trace taken of this phenomenon. The second and fourth modes are simultaneously excited parametrically, and the growth in the fourth mode leads to the eventual participation of the first and third modes through internal resonance. This is a remarkable situation in which excitation at an apparently harmless frequency, far removed from any natural frequency, results in large amplitude responses of all the modes of the structure. Figure 2.3.7 gives a more detailed picture of the 'quasi-steady state' trace obtained for this situation.

Again the external tuning is critical and slight detuning about the central frequency causes erratic behaviour of a cyclic nature.

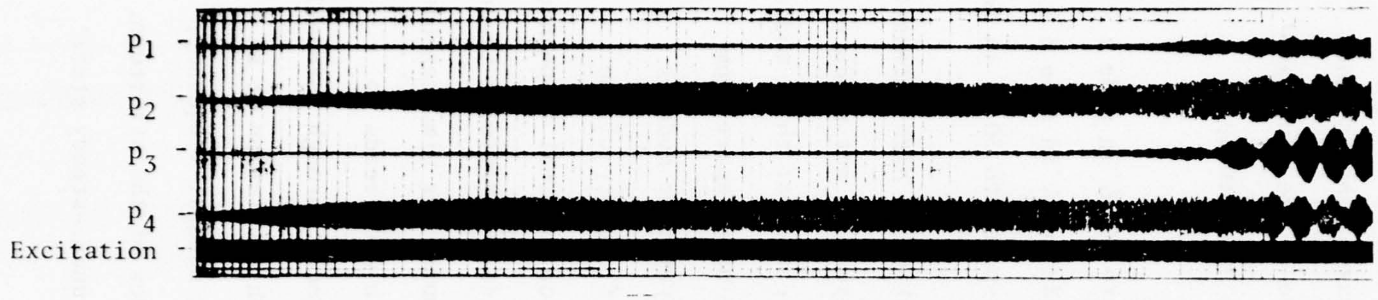


Fig. 2.3.6. Combination Excitation at $\omega_2 + \omega_4$

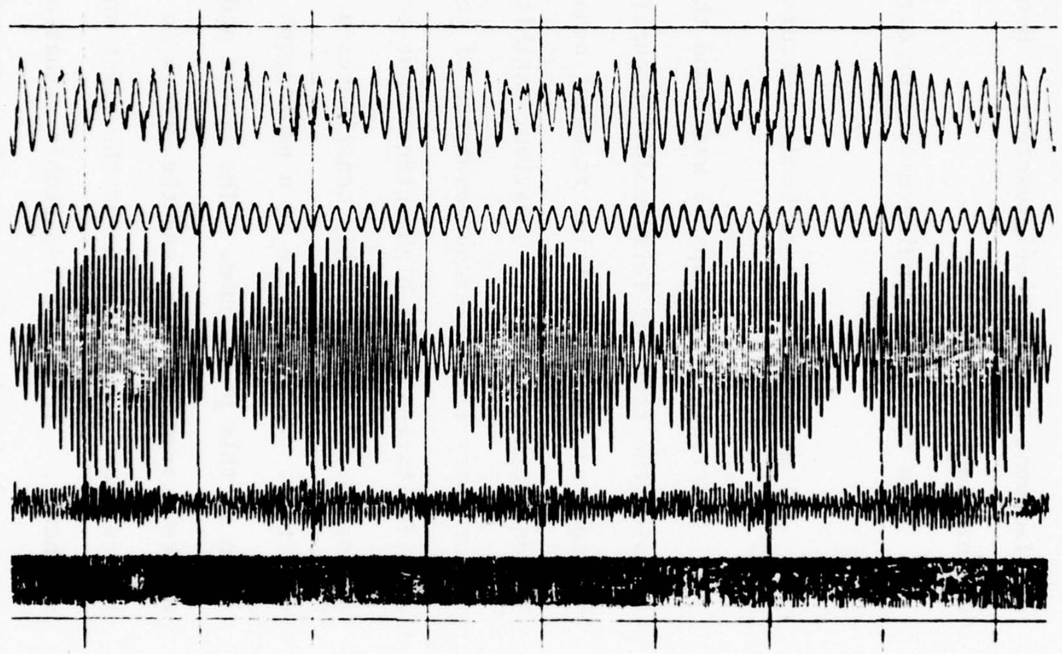


Fig. 2.3.7. 'Quasi' Steady-State

Typically, after initial growth and interaction similar to the tuned case the 'beating' behaviour degenerates and decays, whereupon the process repeats itself.

2.4 RESPONSE FREQUENCIES

For the purposes of the development of the theoretical analysis it was required to determine, where possible, the actual response frequencies of each mode at each external resonance condition, and to find the relationship of these frequencies to the excitation frequency. The frequencies could only be accurately measured for the cases in which steady state is attained, the results of which may be summarised as follows.

For the two mode system under excitation near a natural frequency, the response of that mode is always at the excitation frequency. Also in all cases considered the higher mode always responds at exactly *twice the frequency* of the lower mode, irrespective of internal or external detuning. In the combination case the response frequencies sum exactly to the excitation frequency, again irrespective of the detuning and excitation amplitude. Thus the response frequencies are 'captured' by the excitation frequency such that a linear relationship with exactly integer coefficients exists between them.

In the four mode system it was possible to collect the data only for the case of direct excitation of the highest mode. Again the forced mode's frequency is at that of the excitation, and the first and third modes' frequencies sum to the excitation frequency. However as there are now three modes involved in the internal relationship, the response frequencies of the first and third modes are not fixed upon specification of an excitation frequency, the only condition being that they sum to it. Experimental evidence suggests that a

first approximation for the frequencies can be obtained by assuming that any detuning is shared between the modes in proportion to their natural frequencies. Thus in this case, the appropriate value for Ω_1 (the response frequency of the first mode) would be given by

$$\Omega_1 = \frac{\omega_1}{\omega_1 + \omega_3}\Omega_f$$

where Ω_f is the forcing frequency.

3. THEORY

3.1 EQUATIONS OF MOTION

The general form of the equation of motion of the rth mode of a discrete structure, retaining quadratic inertial nonlinearities and linear parametric terms is

$$\ddot{p}_r + w_r^2 p_r = F_r \cos\Omega_f t - \epsilon[K_{rj} p_j \cos\Omega_f t + \ell_{rij} p_i \ddot{p}_j + m_{rij} \dot{p}_i \dot{p}_j + \omega_r d_{rr} \dot{p}_r] \tag{3.1.1}$$

where summation on the repeated indices i, j only is implied. Examples of the derivation of these equations for specific structural configurations are given in the next section. F_r is the direct forcing term, K_{rj} the parametric term, both at frequencies Ω_f, while ℓ_{rij}, m_{rij} are the quadratic inertial nonlinear terms, ϵ being a small parameter. As in [1] it is assumed for simplicity that the damping is viscous and that there is no damping coupling between the modes.

3.2 SOLUTION TECHNIQUE

A solution is taken from the form

$$p_r = A_{ro}(t) + A_r(t)\cos(\Omega_r t + \phi_r(t)) + f_r \cos(\Omega_f t + \psi_r) + \epsilon a_r(t) + \epsilon^2 b_r(t) \,.. \tag{3.2.1}$$

The first three terms are the principle part of the solution, while the additive terms in powers of ϵ allow for a perturbational treatment. The first term is a D.C. or rectified component, the second a component with slowly varying amplitude and phase at frequency $\Omega_r (\simeq \omega_r)$, the response frequency of the r^{th} mode, as yet unknown. The third term is the component of forced response of the mode at the forcing frequency.

The linear undamped response

$$f_r = F_r/(\omega_r^2 - \Omega_f^2) \text{ and } \psi_r = 0$$

is generally assumed for this component. The effect of the damping

factor d_{rr} on f_r and the phase ψ_r, can be incorporated but this is not usually necessary as the f_r component is brought in only for excitation frequencies Ω_f reasonably well removed from ordinary resonance and at such points the damping plays little part in the response.

The solution form 3.2.1 requires certain modifications when the excitation frequency is at or near a natural frequency of the system. When exciting near the j^{th} natural frequency, $\Omega_f = \Omega_j$ (the mode responds at the forcing frequency) and there is duplication in the response components. Hence the linear forced response of the j^{th} mode is not incorporated in the solution. In such a situation, the excitation will generally be 'soft'. That is, very small excitation amplitudes are required to generate large responses. Thus the F_r's can be taken to be of order ϵ, or less, and so in a first approximation it is reasonable to neglect the linear forced response of all the other modes as well. Also it has been found in such cases that the D.C. terms have little influence on the accuracy of solution. Hence the solution from when the excitation is 'soft' reduces to:

$$p_r = A_r(t)\cos(\Omega_r t + \phi_r(t)) + \epsilon a_r(t) + \epsilon^2 b_r(t) + \ldots \qquad 3.2.3$$

which is similar to that used by Barr et al [1], [3], [4], [5].

However, when the excitation is 'hard', i.e. when it is away from a natural frequency and at an amplitude that can no longer be taken to be small, it has been found that both the linear forced response and the D.C. components are essential to the analysis if reasonable accuracy of this first order solution is to be maintained.

Taking the most general case ('hard' forcing), the solution procedure begins by the substitution of 3.2.1 into 3.1.1. Restriction of the analysis to one involving terms of the first order in ϵ results in:

$$
\begin{aligned}
&\ddot{A}_{ro} + [\ddot{A}_r - A_r(\Omega_r + \dot{\phi}_r)^2]\cos(\Omega_r t + \phi_r) - [A_r\ddot{\phi}_r + 2\dot{A}_r(\Omega_r + \dot{\phi}_r)]\sin(\Omega_r t + \phi_r) - \\
&- f_r\Omega_f^2\cos(\Omega_f t + \psi_r) + \epsilon\ddot{a}_r + \Omega_r^2[A_{ro} + A_r\cos(\Omega_r t + \phi_r) + f_r\cos(\Omega_f t + \psi_r) + \\
&+ \epsilon a_r] = (\Omega_r^2 - \omega_r^2)[A_{ro} + A_r\cos(\Omega_r t + \phi_r) + f_r\cos(\Omega_f t + \psi_r) + \epsilon a_r] + \\
&+ F_r\cos\Omega_f t - \epsilon\{\kappa_{rj}[A_{jo} + A_j\cos(\Omega_j t + \phi_j) + f_j\cos(\Omega_f t + \psi_j)]\cos\Omega_f t + \\
&+ \ell_{rij}[A_{io} + A_i\cos(\Omega_i t + \phi_i) + f_i\cos(\Omega_f t + \psi_i)].[\ddot{A}_{jo} + (\ddot{A}_j - A_j(\Omega_j + \dot{\phi}_j)^2). \\
&.\cos(\Omega_j t + \phi_j) - (A_j\ddot{\phi}_j + 2\dot{A}_j(\Omega_j + \dot{\phi}_j))\sin(\Omega_j t + \phi_j) - f_j\Omega_f^2\cos(\Omega_f t + \psi_j)] + \\
&+ m_{rij}[\dot{A}_{io} + \dot{A}_i\cos(\Omega_i t + \phi_i) - A_i(\Omega_i + \dot{\phi}_i)\sin(\Omega_i t + \phi_i) - f_i\Omega_f\sin(\Omega_f t + \psi_i)]. \\
&.[\dot{A}_{jo} + \dot{A}_j\cos(\Omega_j t + \phi_j) - A_j(\Omega_j + \dot{\phi}_j)\sin(\Omega_j t + \phi_j) - f_j\Omega_f\sin(\Omega_f t + \psi_j)] + \\
&+ \omega_r d_{rr}[\dot{A}_{ro} + \dot{A}_r\cos(\Omega_r t + \phi_r) - A_r(\Omega_r + \dot{\phi}_r)\sin(\Omega_r t + \phi_r) - f_r\Omega_f\sin(\Omega_f t + \psi_r)]\}
\end{aligned}
\tag{3.2.4}
$$

In order to make the analysis manageable at this stage these equations are not treated precisely, and the slowly varying accelerations and products of velocities are neglected ($\ddot{A}_r$, $\ddot{\phi}_r$, $\ddot{A}_{ro}$, $\dot{A}_r\dot{\phi}_r$ etc.). Similarly, the slowly varying velocities of order ε are also neglected.

The analysis continues by considering terms of equal powers in ϵ. The zero order terms form the so called 'variational equations'. Terms which cause resonance in the first order perturbational equation (ϵ^1) are removed and considered along with the variational equations. Hence the resulting 'variational' equations contain the essential elements that control the behaviour of the system. The D.C. terms and the cos and sin terms in $(\Omega_r t + \phi_r)$ are considered separately. In the first approximation the technique is essentially a limited harmonic balance with preferential selection and retention of the more important nonlinear terms.

From 3.2.4 the fundamental variational equations are:

$$A_{ro}\omega_r^2 \;=\; 0 \qquad \text{(D.C. terms)} \qquad 3.2.5$$

$$-2A_r\Omega_r\dot{\phi}_r \;=\; (\Omega_r^2 - \omega_r^2)A_r \qquad (\cos(\Omega_r t + \phi_r)\text{ terms}) \qquad (3.2.6$$

$$-2\dot{A}_r\Omega_r \;=\; 0 \qquad (\sin(\Omega_r t + \phi_r)\text{ terms} \qquad 3.2.7$$

and the first order perturbational equations are:

$$\ddot{a}_r + \omega_r^2 a_r = -\{\kappa_{rj}[A_{jo} + A_j\cos(\Omega_j t + \phi_j) + f_j\cos(\Omega_f t + \psi_j)]\cos\Omega_f t +$$

$$+ \ell_{rij}[A_{io} + A_i\cos(\Omega_i t + \phi_i) + f_i\cos(\Omega_f t + \psi_i)].[-A_j\Omega_j^2\cos(\Omega_j t + \phi_j) - f_j$$

$$\Omega_f^2\cos(\Omega_f t + \psi_j)] + m_{rij}[-A_i\Omega_i\sin(\Omega_i t + \phi_i) - f_i\Omega_f\sin(\Omega_f t + \psi_i)].$$

$$.[-A_j\Omega_j\sin(\Omega_j t + \phi_j) - f_j\Omega_f\sin(\Omega_f t + \psi_j)] + \omega_r d_{rr}[-A_r\Omega_r\sin(\Omega_r t$$

$$+ \phi_r) - f_r\Omega_f\sin(\Omega_f t + \psi_r)]\}$$

3.2.8

It must be noted that when the forcing is 'soft', the F_r term (order ϵ) should also be included in this equation as it will be a resonant term for the mode that is being directly excited. However in that case the D.C. terms (A_{ro}), and the linear forced response terms (f_r) will be omitted.

The first damping term in 3.2.8 (second last term on RHS of the equation) is clearly resonant and must be removed to the corresponding variational equation. D.C. terms arise where there is a product of two trigonometric functions having the same frequency. For example

$\cos(\Omega_f t + \psi_j)\cos\Omega_f t$ expands to $\frac{1}{2}[\cos(2\Omega_f t + \psi_j) + \cos\psi_j]$, a harmonic part at $2\Omega_f$ and a D.C, part (ψ_j constant for a given excitation frequency). All such D.C. terms are removed to the D.C. variational equations. Other terms can become resonant if there are certain relationships between the forcing frequency and the responding frequencies, and amongst the responding (near natural) frequencies themselves.

3.3 EXTERNAL AND INTERNAL RESONANCE

On examination, equations 3.2.8 yield the frequency relationships that may cause resonance in the system. Consider the parametric term, $\kappa_{rj}(\ldots)$, then as $\cos\Omega_f t\cos(\Omega_j t + \phi_j)$ expands to $\frac{1}{2}[\cos(\Omega_f t + \Omega_j t + \phi_j) + \cos(\Omega_f t - \Omega_j t - \phi_j)]$, this term can be resonant in the j^{th} mode if $\Omega_f = 2\Omega_j$ or in the i^{th} mode if $\Omega_f = \Omega_i \pm \Omega_j$. These frequency relations are between the excitation frequency and either one or two of the modal response frequencies and hence are termed 'external' resonance conditions. These are in fact the ordinary parametric and combination parametric resonance conditions. The terms involving products of the excitation and response frequencies in the ℓ_{rij} and m_{rij} 'nonlinear' brackets of 3.3.8 also yield the same conditions. Given the external resonance condition these are the types of terms that may cause external excitation of the modes involved in it.

The nonlinear terms, however, predict a further type of resonance condition. Terms involving products such as $\cos(\Omega_i t + \phi_i)\cos(\Omega_j t + \phi_j)$ produce fundamental harmonics of $(\Omega_i \pm \Omega_j)$. These can be resonant in the j^{th} mode if $\Omega_i = 2\Omega_j$ or in the r^{th} mode if $\Omega_r = \Omega_i \pm \Omega_j$. These conditions are termed 'internal' resonance conditions as they are independent of the external excitation but depend upon the natural

frequencies of the system. Given an internal resonance condition, the nonlinear terms may be such as to allow a mode that is already excited to excite another of the modes involved in the 'internal' relationship (otherwise known as 'autoparametric' interaction).

To sum up, the possible resonance conditions are:-

$$\left.\begin{array}{lll} \Omega_f = \Omega_i & \text{(direct 'soft' excitation)} \\ \Omega_f = 2\Omega_i & \text{('parametric' excitation)} \\ \Omega_f = \Omega_i \pm \Omega_j & \text{('combination' parametric excitation)} \end{array}\right\} \text{external resonance}$$

$$\left.\begin{array}{lll} \Omega_i = 2\Omega_j & \text{(implying } \omega_i \simeq 2\omega_j) \\ \Omega_r = \Omega_i + \Omega_j & \text{(implying } \omega_r \simeq \omega_i + \omega_j) \end{array}\right\} \text{internal resonance}$$

Thus, to the order ϵ^1, one external forcing frequency can excite up to a maximum of two modes. It is possible that each of these excited modes in turn can, if the internal resonance conditions exist, excite up to two more modes 'autoparametrically', and so on. Experimental evidence of this cascading effect has been presented in Section 2.

Cubic and higher order nonlinearities give rise to other external [6] and internal [1] resonance conditions. For example cubics can give external conditions such as $\Omega_f = \Omega_i + 2\Omega_j$, $\Omega_i + \Omega_j + \Omega_k$, and internal conditions such as $\Omega_r = 2\Omega_i + \Omega_j$, $\Omega_i + \Omega_j + \Omega_k$. The higher the order of nonlinearity the more modes can become involved. Here interest is confined to the quadratic nonlinearities and the parametric terms only.

3.4 THE VARIATIONAL EQUATIONS

At this stage in the analysis it becomes necessary to study particular structural systems. Interest is focussed on systems having

internal resonance of which for this class there are two types, as shown in the previous section. They involve either two or three modes such that

$$2\Omega_1 = \Omega_2 \quad \text{or} \quad \Omega_1 + \Omega_2 = \Omega_3 \quad (\Omega_1 < \Omega_2 < \Omega_3)$$

It is of course possible for both these conditions to hold simultaneously, but this situation is less likely to be encountered with in practice.

For a system having internal resonance between two modes in the form $\Omega_2 = 2\Omega_1$, the external resonance possibilities are:

$$\Omega_f = \Omega_1, \Omega_2, 2\Omega_2, \Omega_2 + \Omega_1$$

where the first two are 'soft' forcing cases (note that $2\Omega_1 = \Omega_2$, and $\Omega_2 - \Omega_1 = \Omega_1$).

For a system with three modes in internal resonance, $\Omega_1 + \Omega_2 = \Omega_3$, the external resonance possibilities are:

$$\Omega_f = \Omega_1, \Omega_2, \Omega_3, 2\Omega_1, 2\Omega_2, 2\Omega_3, \Omega_2 - \Omega_1, \Omega_3 + \Omega_2, \Omega_3 + \Omega_1,$$

where the first three are 'soft' forcing cases (note that $\Omega_2 + \Omega_1 = \Omega_3$, $\Omega_3 - \Omega_2 = \Omega_1$, $\Omega_3 - \Omega_1 = \Omega_2$)

Each of these external resonance conditions produces a corresponding set of variational equations. In each case these are derived by removal of all the resonant terms from 3.3.8 into the appropriate fundamental variational equation, 3.3.5, 6, 7. The two mode systems are examined first.

(1) <u>Two mode systems</u>

Taking the first case, $\Omega_f = \Omega_1$ with $\Omega_2 = 2\Omega_1$, and neglecting in the solution form the components of linear forced response and the D.C. terms, ('soft' forcing, see equation 3.2.3), the resonant terms for mode 1 are those involving

$$\kappa_{12},\ \ell_{112},\ \ell_{121},\ m_{112},\ m_{121},\ \omega_1 d_{11} \text{ and } F_1,$$

and for mode 2:-

$$\kappa_{21},\ \ell_{211},\ m_{211},\ \omega_2 d_{22}.$$

The resulting variational equations are:-

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 + F_1\cos\phi_1 - \frac{\epsilon}{2}[P_1A_2\cos(\phi_2 - \phi_1) + N_1A_1A_2\cos(\phi_2 - 2\phi_1)]$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-P_1A_2\sin(\phi_2 - \phi_1) - N_1A_1A_2\sin(\phi_2 - 2\phi_1) - 2\omega_1\Omega_1 d_{11}A_1] + F_1\sin\phi$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_1\cos(\phi_2 - \phi_1) - N_2A_1^2\cos(\phi_2 - 2\phi_1)]$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_1\sin(\phi_2 - \phi_1) - N_2A_1^2\sin(\phi_2 - 2\phi_1) - 2\omega_2\Omega_2 d_{22}A_2]$$

where $P_1 = \kappa_{12}$
$N_1 = (m_{112} + m_{121})\Omega_1\Omega_2 - \ell_{112}\Omega_2^2 - \ell_{121}\Omega_1^2$
$P_2 = \kappa_{21}$
$N_2 = (\ell_{211} + m_{211})\Omega_1^2$ 3.4.1

Similarly for $\Omega_f = \Omega_2$ with $\Omega_2 = 2\Omega_1$, the resulting variational equations are:-

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_1\cos 2\phi_1 + N_1A_1A_2\cos(\phi_2 - 2\phi_1)]$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[P_1A_1\sin 2\phi_1 - N_1A_1A_2\sin(\phi_2 - 2\phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 + F_2\cos\phi_2 - \frac{\epsilon}{2}[-N_2A_1^2\cos(\phi_2 - 2\phi_1)]$$

$$-2\dot{A}_2\Omega_2 = F_2\sin\phi_2 - \frac{\epsilon}{2}[-N_2A_1^2\sin(\phi_2 - 2\phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

where $P_1 = \kappa_{11}$, and N_1 and N_2 are as in 3.4.1 3.4.2

For the hard forcing cases, $\Omega_f = \Omega_1 + \Omega_2$ and $2\Omega_2$, the equations are considerably more complex, as the linear forced response and the D.C. terms are included in the solution form. For the two mode system the D.C. 'variational equations' are:-

$$A_{10}\omega_1^2 = -\frac{\epsilon}{2}\ [\kappa_{11}f_1 + \kappa_{12}f_2 + (m_{112} + m_{121} - \ell_{112} - \ell_{121})f_1f_2\Omega_f^2$$

$$+ (m_{11} - \ell_{111})f_1^2\Omega_f^2 + (m_{122} - \ell_{122})f_2^2\Omega_f^2$$

$$+ (m_{111} - \ell_{111})A_1^2\Omega_1^2 + (m_{122} - \ell_{122})A_2^2\Omega_2^2\,]$$

$$A_{20}\omega_2^2 = -\frac{\epsilon}{2}\ [\kappa_{21}f_1 + \kappa_{22}f_2 + (m_{221} + m_{212} - \ell_{212} - \ell_{221})f_1f_2\Omega_f^2$$

$$+ (m_{211} - \ell_{211})f_1^2\Omega_f^2 + (m_{222} - \ell_{222})f_2^2\Omega_f^2$$

$$+ (m_{211} - \ell_{211})A_1^2\Omega_1^2 + (m_{222} - \ell_{222})A_2^2\Omega_2^2\,] \qquad 3.4.3$$

The derivation of these equations is not affected by the internal and external frequency relationships, and so they are the same for both the hard forcing cases.

For the combination external resonance case, $\Omega_f = \Omega_1 + \Omega_2$ with $\Omega_2 = 2\Omega_1$, the slowly varying amplitude and phase variational equations, with $f_r = \dfrac{F_r}{(\omega_r^2 - \Omega_f^2)}$ and $\psi_r = 0$, are:

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_2\cos(\phi_1 + \phi_2) + N_1A_1A_2\cos(\phi_2 - 2\phi_1) -$$

$$-2(\ell_{111}A_{10} + \ell_{121}A_{20})A_1\Omega_1^2]$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[P_1A_2\sin(\phi_1 + \phi_2) - N_1A_1A_2\sin(\phi_2 - 2\phi_1) - 2\omega_1\Omega_1d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_1\cos(\phi_1 + \phi_2) - N_2A_1^2\cos(\phi_2 - 2\phi_1) -$$

$$-2(\ell_{212}A_{10} + \ell_{222}A_{20})A_2\Omega_2^2]$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_1\sin(\phi_1 + \phi_2) - N_2A_1^2\sin(\phi_2 - 2\phi_1) - 2\omega_2\Omega_2d_{22}A_2]$$

where $P_1 = \kappa_{12} - \ell_{112}f_1\Omega_2^2 - \ell_{121}f_1\Omega_f^2 - \ell_{122}f_2(\Omega_2^2 + \Omega_f^2) + 2m_{122}\cdot$

$\cdot f_2\Omega_2\Omega_f + (m_{112} + m_{121})f_1\Omega_2\Omega_f$

$P_2 = \kappa_{21} - \ell_{212}f_2\Omega_f^2 - \ell_{221}f_2\Omega_1^2 - \ell_{211}f_1(\Omega_1^2 + \Omega_f^2) + 2m_{211}\cdot$

$\cdot f_1\Omega_1\Omega_f + (m_{221} + m_{212})f_2\Omega_1\Omega_f$

N_1 and N_2 are as in 3.4.1. 3.4.4

It is obvious that certain similarities exist between equations 3.4.1, 2 and 4. The terms involving F_4 and P_r are those which arise from the external resonance condition. Terms involving N_r are those which arise from the internal resonance condition, and these are identical in each set. In the hard forcing situation the parametric (P_r) type terms not only contain the ordinary parametric terms (κ_{rj}) but also nonlinear terms arising from the inclusion of the linear forced response components in the solution.

The D.C. 'variational' equations 3.4.3 can, by inspection of 3.2.8, be written in a general form applicable to an n mode structure:-

$$A_{ro}\omega_r^2 = -\frac{\epsilon}{2}[\kappa_{ri}f_i + (m_{rij} - \ell_{rij})f_i f_j \Omega_f^2 + (m_{rjj} - \ell_{rjj})A_j^2\Omega_j^2]$$

$$i,j = 1,\ldots, n \quad \text{(no sum on r)} \tag{3.4.5}$$

Also by inspection of 3.4.5, the additions that these D.C. terms make to the slowly varying variational equations can be written in general form. The addition to the right-hand-side of each $-2A_r\Omega_r\dot{\phi}_r$ equation takes the form

$$\nabla_r = \sum_{i=1}^{n} \epsilon \ell_{rir} A_{io} A_r \Omega_r^2 \tag{3.4.6}$$

Equations 3.4.5 and 6 cover the D.C. effects in all the cases to be considered.

The variational equations for the final two mode system, $\Omega_f = 2\Omega_2$ with $\Omega_2 = 2\Omega_1$, are:

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[N_1A_1A_2\cos(\phi_2 - 2\phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-N_1A_1A_2\sin(\phi_2 - 2\phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_2\cos 2\phi_2 - N_2A_1^2\cos(\phi_2 - 2\phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_2\sin 2\phi_2 - N_2A_1^2\sin(\phi_2 - 2\phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

where $P_2 = \kappa_{22} - f_1(\ell_{221}\Omega_f^2 + \ell_{212}\Omega_2^2 - (m_{221} + m_{212})\Omega_2\Omega_f) -$

$$-f_2(\ell_{222}(\Omega_f^2 + \Omega_2^2) - 2m_{222}\Omega_2\Omega_f)$$

and N_1 and N_2 are as in 3.4.1. 3.4.7

In all of these systems of equations (3.4.1, 2, 4 and 7), the terms arising from the internal resonance condition, the N_r terms, are the same. If the system possesses no such internal resonance condition then these terms simply disappear, leaving equations similar in form to those studies by Yamamoto et al [6, 7].

(2) Three Mode Systems

There are nine separate external resonance cases for the three mode system. Only the ones to be used subsequently are listed here. All the other cases are listed in Appendix I.

$\underline{\Omega_f = \Omega_3}$, $\underline{\Omega_1 + \Omega_2 = \Omega_3}$ ('soft' forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_2\cos(\phi_1 + \phi_2) + N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[P_1A_2\sin(\phi_1 + \phi_2) - N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_1\cos(\phi_1 + \phi_2) + N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_1\sin(\phi_1 + \phi_2) - N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 + F_3\cos\phi_3 - \frac{\epsilon}{2}[N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_3\Omega_3 = F_3\sin\phi_3 - \frac{\epsilon}{2}[N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3 d_{33}A_3]$$

where $P_1 = \kappa_{12}$

$P_2 = \kappa_{21}$

$N_1 = (m_{123} + m_{132})\Omega_2\Omega_3 - \ell_{123}\Omega_3^2 - \ell_{132}\Omega_2^2$

$N_2 = (m_{213} + m_{231})\Omega_1\Omega_3 - \ell_{231}\Omega_1^2 - \ell_{213}\Omega_3^2$

$N_3 = -(m_{312} + m_{321})\Omega_1\Omega_2 - \ell_{312}\Omega_2^2 - \ell_{321}\Omega_1^2$

3.4.8

From these equations, including those in Appendix I, it is possible to deduce the equations for any given multimode system. For example, the equations for a system having four modes and resonant frequency relationships of:

$\underline{\Omega_f = \Omega_2 + \Omega_4}$ with $\underline{\Omega_1 + \Omega_3 = \Omega_4}$ are:

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[N_1A_3A_4\cos(\phi_4 - \phi_3 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-N_1A_3A_4\sin(\phi_4 - \phi_3 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_4\cos(\phi_2 + \phi_4)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_4\sin(\phi_2 + \phi_4) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[N_3A_1A_4\cos(\phi_4 - \phi_3 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[-N_3A_1A_4\sin(\phi_4 - \phi_3 - \phi_1)]$$

$$-2A_4\Omega_4\dot{\phi}_4 = (\Omega_4^2 - \omega_4^2)A_4 - \frac{\epsilon}{2}[P_4A_2\cos(\phi_2 + \phi_4) + N_4A_1A_3\cos(\phi_4 - \phi_3 - \phi_1)] + \nabla_4$$

$$-2\dot{A}_4\Omega_4 = -\frac{\epsilon}{2}[P_4A_2\cos(\phi_2 + \phi_4) + N_4A_1A_3\sin(\phi_4 - \phi_3 - \phi_1) - 2\Omega_4\omega_4 d_{44}A_4]$$

where $P_2 = \kappa_{24} - \ell_{244}f_4(\Omega_4^2 + \Omega_f^2) - f_1(\ell_{214}\Omega_4^2 - \ell_{241}\Omega_f^2) - f_2(\ell_{224}\Omega_4^2 +$

$+ \ell_{242}\Omega_f^2) - f_3(\ell_{234}\Omega_4^2 - \ell_{243}\Omega_f^2) + 2m_{244}f_4\Omega_4\Omega_f + (m_{214} + m_{241})f_1\Omega_4\Omega_f +$

$+ (m_{242} + m_{224})f_2\Omega_4\Omega_f + (m_{234} + m_{243})f_3\Omega_4\Omega_f$

and $P_4 = \kappa_{42} - \ell_{422}f_2(\Omega_2^2 + \Omega_f^2) - f_1(\ell_{412}\Omega_2^2 - \ell_{421}\Omega_f^2) - f_3(\ell_{432}\Omega_2^2 +$

$+ \ell_{423}\Omega_f^2) - f_4(\ell_{442}\Omega_2^2 + \ell_{424}\Omega_f^2) + 2m_{422}f_2\Omega_2\Omega_f + (m_{412} + m_{421})f_1\Omega_2\Omega_f +$

$+ (m_{432} + m_{423})f_3\Omega_2\Omega_f + (m_{442} + m_{424})f_4\Omega_2\Omega_f.$

and

and

$$N_1 = (m_{134} + m_{143})\Omega_3\Omega_4 - \ell_{134}\Omega_4^2 - \ell_{143}\Omega_3^2$$
$$N_3 = (m_{314} + m_{341})\Omega_1\Omega_4 - \ell_{314}\Omega_4^2 - \ell_{341}\Omega_1^2$$
$$N_4 = -(m_{413} + m_{431})\Omega_1\Omega_3 - \ell_{413}\Omega_3^2 - \ell_{431}\Omega_1^2. \qquad 3.4.9$$

3.5 SOLUTION OF THE VARIATIONAL EQUATIONS

The standard procedure at this stage is to seek steady state, or stationary solutions by setting $\dot{A}$ and $\dot{\phi}$ to zero [B3, H2, I1, A1, Y2]. However, experimental evidence shows that only in certain cases will stationary conditions be attained. More generally there is some form of continuous 'heating' between the modes involved.

Yamamoto and Saito [7], when considering purely combination parametric systems which exhibit exponential growth in an instability region, assume 'normal' solutions by taking $\dot{\phi} = 0$. This results in a relationship giving the unknown response frequencies, Ω_r, in terms of the excitation and natural frequencies, Ω_f and ω_r. In other words it specifies the 'detuning' of the system defining by how much the response frequencies differ from the linear natural frequencies. In the event, even for the simple systems considered, the relationship is not trivial and an approximation is employed to simplify it before the continuation of analysis.

In an allied paper Yamamoto and Hayashi [6] consider two mode systems having quadratic and cubic elastic nonlinearities under harmonic excitation. The external resonance cases studied are $\Omega_f = 2\Omega_1 + \Omega_2$ and $\Omega_f = \Omega_1 + \Omega_2$. Each resulting variational equation contains only one trigonometric function (because of the lack of internal resonance and parametric terms), and so analytical solutions are obtained upon setting $\dot{A}$ and $\dot{\phi}$ to zero. Again the detuning relationships are quite complex, but for a first approximation

neglecting higher order terms, an estimate for the response frequencies is given by:

$$\frac{\Omega_r}{\Omega_f} = \frac{\omega_r}{\omega_1 + \omega_2} \qquad \text{for } \Omega_f = \Omega_1 + \Omega_2 \text{ case}, \quad r = 1,2$$

$$\frac{\Omega_r}{\Omega_f} = \frac{\omega_r}{2\omega_1 + \omega_2} \qquad \text{for } \Omega_f = 2\Omega_1 + \Omega_2 \text{ case}, \quad r = 1,2 \tag{3.5.1}$$

The ratio of response frequency to excitation frequency is equal to the ratio of natural frequency to the corresponding 'tuned' exciting frequency. In other words the 'external' detuning is shared proportionately between the modes. Experimental evidence suggests that this is a reasonable approximation.

Where there are both external and internal resonance conditions the situation becomes more complex. For two mode systems the analysis procedure has already set the response frequencies exactly. For example 3.4.2 has been derived by taking $\Omega_f = \Omega_2$ with $\Omega_2 = 2\Omega_1$. For a given excitation frequency the response frequencies are fixed. In the combination case, $\Omega_f = \Omega_1 + \Omega_2$ but $\Omega_2 = 2\Omega_1$ and so again Ω_1 and Ω_2 are fixed. These frequency relationships are confirmed experimentally.

For three mode systems with soft forcing, the response frequency of the directly resonant mode is fixed at the excitation frequency, but the only other condition on the other modes involved through the internal resonance relationship is that $\Omega_i + \Omega_j = \Omega_f$, and so the frequencies Ω_i and Ω_j are not fixed exactly. In the hard forcing combination situation none of the frequencies are set exactly through the solution procedure (e.g. $\Omega_f = \Omega_2 + \Omega_3$, $\Omega_1 + \Omega_2 = \Omega_3$).

It has not been found possible to obtain general analytical solutions to any of the sets of complete variational equations listed in Section 2.5 and Appendix I.

As they do not lend themselves to analytical solution, recourse has to be made to numerical solution for particular structural systems. Where necessary, equations of the type 3.5.1 are used to give values for unfixed frequency. Where steady state conditions are found to exist, by numerical integration, then it is possible to compute frequency response curves by numerical solution of the nonlinear algebraic equations that result from setting $\dot{A}_r$ and $\dot{\phi}_r$ to zero. The accuracy of the solution procedure can be checked, in the simpler cases at least, by direct integration of the full equations of motion under the same excitation conditions.

Analytically, more progress can be achieved if certain assumptions are made. In the soft forcing cases analytical solutions can be found if the 'parametric' terms are neglected. This gives results similar to those presented by Barr and Nelson [1], these will not be discussed further here. In the hard forcing situations the initial resonant behaviour is governed by the 'parametric' type terms in the variational equations. The terms arising from the internal resonance conditions, or 'autoparametric' terms only become dominant once growth in a relevant mode has begun. Hence by neglect of these terms it is possible to derive expressions which will give the approximate stability boundaries for the system. Recourse has to be made to the full variational equations in order to study the resonant behaviour of each mode involved.

3.6 STABILITY BOUNDARIES - hard forcing cases

For vanishingly small $A_{i,j}$, allowing the neglect of the auto-parametric coupling terms in the complete variational equations, there are regions in the excitation amplitude/frequency parameter space for these cases inside which the $A_{i,j}$ are subject to exponential growth with time. In this section the boundary of this region, which is effectively the boundary of the region inside which growth and interaction occur, is found. The external resonance conditions under consideration are:

$$\Omega_f = \Omega_j \pm \Omega_i \quad \text{or} \quad \Omega_f = 2\Omega_i. \quad (j > i,\ i \neq j)$$

Taking the first of these and neglecting the autoparametric terms, the variational equations for the i^{th} and j^{th} modes of an n mode structure are:-

$$-2A_i\Omega_i\dot{\phi}_i = (\Omega_i^2 - \omega_i^2)A_i - \frac{\epsilon}{2} P_i A_j \cos(\phi_j \pm \phi_i) + \sum_{k=1}^{n} \epsilon \ell_{iki} A_{ko} A_i \Omega_i^2 \tag{1}$$

$$-2\dot{A}_i\Omega_i = \mp \frac{\epsilon}{2} P_i A_j \sin(\phi_j \pm \phi_i) + \epsilon\Omega_i\omega_i d_{ii} A_i \tag{2}$$

$$-2A_j\Omega_j\dot{\phi}_j = (\Omega_j^2 - \omega_j^2)A_j - \frac{\epsilon}{2} P_j A_i \cos(\phi_j \pm \phi_i) + \sum_{k=1}^{n} \epsilon \ell_{jkj} A_{ko} A_j \Omega_j^2 \tag{3}$$

$$-2\dot{A}_j\Omega_j = - \frac{\epsilon}{2} P_j A_i \sin(\phi_j \pm \phi_i) + \epsilon\Omega_j\omega_j d_{jj} A_j \tag{4}$$

where $P_i = \kappa_{ij} + \sum_{k=1}^{n} [-\ell_{ijk} f_k \Omega_f^2 - \ell_{ikj} f_k \Omega_j^2 + (m_{ijk} + m_{ikj}) f_k \Omega_f \Omega_j]$

and $P_j = \kappa_{ji} + \sum_{k=1}^{n} [-\ell_{jik} f_k \Omega_f^2 - \ell_{jki} f_k \Omega_i^2 \pm (m_{jik} + m_{jki}) f_k \Omega_i \Omega_f]$

These can be deduced from equations I6, 7, and 8, the upper and lower signs being adopted for summed and difference combination conditions respectively. The D.C. terms are given by 3.4.5, which here take the reduced from

$$A_{ko}\omega_k^2 = -\frac{\epsilon}{2}\left[\kappa_{kp}f_p + (m_{kpq} - \ell_{kpq})f_p f_q \Omega_f^2\right]$$

where $p,q = 1,n$ 3.6.2

as the terms involving the response amplitudes, A_j^2, are omitted zero at the onset of instability).

Let

$$\Delta_i = \Omega_i^2 - \omega_i^2 + \sum_{k=1}^{n} \epsilon\ell_{iki}A_{ko}\Omega_i^2 \qquad 3.6.3$$

Then taking a solution of the form

$$\dot{\phi}_{i,j} = 0 \qquad 3.6.4$$

equations 3.6.1(1) and (3) yield

$$\frac{A_i^2}{A_j^2} = \frac{\Delta_j P_i}{\Delta_i P_j} \qquad 3.6.5$$

$$\text{and } \Delta_i\Delta_j = \frac{\epsilon^2}{4} P_i P_j \cos^2(\phi_j \pm \phi_i) \qquad 3.6.6$$

Hence the amplitude ratios are fixed, implying common growth rates.

Let

$$A_{i,j} = a_{i,j}e^{\alpha t} \qquad 3.6.7$$

then 3.6.1(2) and (4) yield:

$$\frac{\epsilon^2 P_i P_j}{4} \sin^2(\phi_j \pm \phi_i) = \pm (2\alpha\Omega_i + \epsilon\Omega_i\omega_i d_{ii})(2\alpha\Omega_j + \epsilon\Omega_j\omega_j d_{jj}) \qquad 3.6.8$$

For P_i, P_j non-zero, 3.6.6 and 8 give

$$\alpha^2 + \frac{\epsilon}{2}(\omega_j d_{jj} + \omega_i d_{ii})\alpha \pm \frac{\Delta_i\Delta_j}{4\Omega_i\Omega_j} + \frac{\epsilon^2}{4}(\omega_i d_{ii}\omega_j d_{jj} - \frac{P_i P_j}{4\Omega_i\Omega_j}) = 0$$

having roots

$$\alpha = -\frac{\epsilon}{4}(\omega_i d_{ii} + \omega_j d_{jj}) \pm \left[\frac{\epsilon^2}{16}(\omega_i d_{ii} - \omega_j d_{jj})^2 \mp \frac{\Delta_i\Delta_j}{4\Omega_j\Omega_j} \pm \frac{\epsilon^2 P_i P_j}{16\Omega_i\Omega_j}\right]^{\frac{1}{2}}$$

3.6.9

The circled ± sign does not refer to summed and difference combination conditions. The critical root as regards stability is the more positive one [Y1], viz.

$$\alpha = -\frac{\epsilon}{4}(\omega_i d_{ii} + \omega_j d_{jj}) + \left[\frac{\epsilon^2}{16}(\omega_i d_{ii} - \omega_j d_{jj})^2 \mp \frac{\Delta_i\Delta_j}{4\Omega_i\Omega_j} \pm \frac{\epsilon^2 P_i P_j}{16\Omega_i\Omega_j}\right]^{\frac{1}{2}}$$

3.6.10

For $\alpha > 0$, growth will occur leading to resonant behaviour. Hence the stability boundary is defined by

$$\pm \epsilon^2 P_i P_j = 4[\epsilon^2\omega_i d_{ii}\omega_j d_{jj}\Omega_i\Omega_j \pm \Delta_i\Delta_j] \qquad 3.6.11$$

the equality being replaced by > for growth.

Again, upper and lower signs are adopted for summed and difference conditions respectively. For a given structure, 3.6.11 is a function of excitation frequency and amplitude. Hence for a given resonance condition it is possible to compute stability diagrams in the excitation amplitude-frequency plane.

The governing equations for the simpler case, when $\Omega_f = 2\Omega_i$, are:

$$-2A_i\Omega_i\dot{\phi}_i = (\Omega_i^2 - \omega_i^2)A_i - \frac{\epsilon}{2}P_iA_i\cos 2\phi_i + \sum_{k=1}^{n} \epsilon \ell_{iki}A_{ko}A_i\Omega_i^2$$

$$-2\dot{A}_i\Omega_i = -\frac{\epsilon}{2}P_iA_i\sin 2\phi_i + \epsilon\Omega_i\omega_i d_{ii}A_i$$

$$\text{where} \quad P_i = \kappa_{ii} + \sum_{k=1}^{n}[-\ell_{iki}f_k\Omega_i^2 - \ell_{iik}f_k\Omega_f^2 + (m_{iik} + m_{iki})f_k\Omega_i\Omega_f]$$

3.6.12

Repetition of the preceding analysis yields the growth condition:

$$\epsilon^2P_i \geqslant 4[\epsilon^2\omega_i^2d_{ii}^2\Omega_i^2 + \Delta_i^2] \qquad 3.6.13$$

It must be noted that the D.C.terms are functions of the response amplitudes, A_r, and when approaching the stability boundaries from within the unstable region the stability boundaries will be affected accordingly. The anlysis presented here is limited to approaching the boundaries from the 'outside' only.

4. STRUCTURAL EXAMPLES

4.1 INTRODUCTION

This section presents a brief report of the analysis and selected numerical results from the theoretical models which are based on the experimental models of Section 2. At each resonance situation the results of the integration of the relevant 'variational' equations are compared with the direct integration of the full equations of motion. In the 'hard' forcing cases examples are given showing the stability boundaries in the excitation amplitude/frequency plane.

4.2 THE TWO MODE SYSTEM

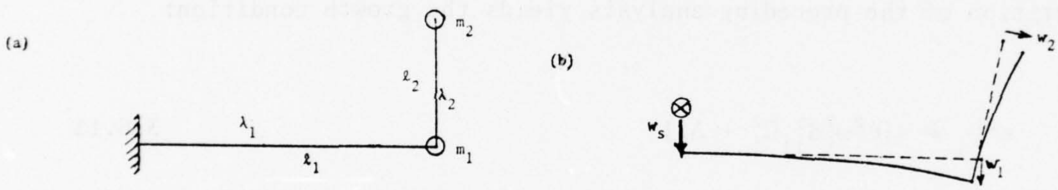


Fig. 4.2.1

By approximating the experimental structure by a two mode lumped mass model with coordinates as shown in Figure 4, the equations of motion can be formulated by application of Lagrange's procedure to the kinetic and potential energy functions. The components of 'axial' motion of each mass (perpendicular to w_1 and w_2) are kinematically related to the end displacements w_1 and w_2 by the assumption of a deflection curve for each section, similar to that of a cantilever beam

under static end load. The resulting equations of motion retaining up to quadratic inertial nonlinearities and linear parametric terms are found to be:

$$[m_1 + m_2(1 + 2.25\frac{\ell_2^2}{\ell_1^2})]\ddot{w}_1 + 1.5\, m_2\frac{\ell_2}{\ell_1}\ddot{w}_2 + \lambda_1 w_1$$

$$= -[m_1 + m_2(1 + 2.25\frac{\ell_2}{\ell_1}w_1 + \frac{1.5}{\ell_1}w_2]\ddot{w}_s -$$

$$-m_2[0.9\frac{\ell_2}{\ell_1^2}w_1\ddot{w}_1 + 0.45\frac{\ell_2}{\ell_1^2}\dot{w}_1^2 + \frac{1.2}{\ell_2}(w_2\ddot{w}_2 + \dot{w}_2^2) + \frac{0.3}{\ell_1}w_1\ddot{w}_2 +$$

$$\frac{3.0}{\ell_1}(\ddot{w}_1 w_2 + \dot{w}_1\dot{w}_2)$$

$$1.5\, m_2\frac{\ell_2}{\ell_1}\ddot{w}_1 + m_2\ddot{w}_2 + \lambda_2\ddot{w}_2$$

$$= -m_2[\frac{1.2}{\ell_2}w_2\ddot{w}_s + \frac{1.5}{\ell_1}(w_1\ddot{w}_s + w_1\ddot{w}_1) - \frac{1.2}{\ell_1}(w_1\ddot{w}_1 + \dot{w}_1^2) +$$

$$+\frac{1.2}{\ell_2}w_2\ddot{w}_1$$

4.2.1

where λ_1 and λ_2 are the static bending stiffnesses at m_1 and m_2 of the cantilevers of length ℓ_1 and ℓ_2 respectively, which make up the model.

4.3 NATURAL FREQUENCIES AND NORMAL MODES - CONVERSION TO NORMAL COORDINATES

The equations of motion 4.2.1 may be written in matrix form

$$[A]\ddot{\underset{\sim}{w}} + [C]\underset{\sim}{w} = \underset{\sim}{f}(w,\dot{w},\ddot{w},t) \qquad 4.3.1$$

where [A] and [C] are the linear symmetric inertia and stiffness matrices

and f is a column vector comprising the direct forcing, parametric and quadratic nonlinear terms.

The homogeneous part of 4.3.1 yields the linear natural frequencies (eigenvalues) and normal modes (eigenvectors), the derivation of which may be found in any standard text [8].

Applying the linear transformation

$$\underset{\sim}{w} = [R]\underset{\sim}{p}$$

to 4.3.1, ([R] being the matrix of eigenvectors, or 'modal' matrix), we have

$$\ddot{\underset{\sim}{p}} + [R]^{-1}[A]^{-1}[C][R]\underset{\sim}{p} = [R]^{-1}[A]^{-1}\underset{\sim}{g}(p,\dot{p},\ddot{p},t) \qquad 4.3.2$$

where $[R]^{-1}[A]^{-1}[C][R] = [\omega^2]$ is the diagonal matrix of eigenvalues.

Numerical parameters are chosen which are roughly based on the experimental system, one value of stiffness being adjusted to give the required internal frequency relationship.

In the following examples the values of the various parameters taken are

$$m_1 = 0.3 \text{ kg}, \quad m_2 = 0.1 \text{ kg}, \quad \ell_1 = 0.15 \text{ m}, \quad \ell_2 = 0.1050 \text{ m}$$

$$\lambda_1 = 2040.221 \text{ N/m}, \quad \lambda_2 = 1.0 \times 10^3 \text{ N/m} \qquad 4.3.3$$

yielding natural frequencies of 119.52 and 59.76 rads/sec.

Viscous damping is also incorporated and again rough estimates based upon experimental data are used suggesting values, $\epsilon d_{11} = .012$,

$\varepsilon_{22} = .024.$

Equations 4.3.2 are equivalent to equations 3.1.1, and the coefficients of the direct forcing, parametric and nonlinear terms in normal coordinates may be computed through 4.2.1 and 4.3.3 via the transformation matrix $[R]^{-1}[A]^{-1}$.

4.4 'SOFT' FORCING RESULTS, TWO MODE SYSTEM

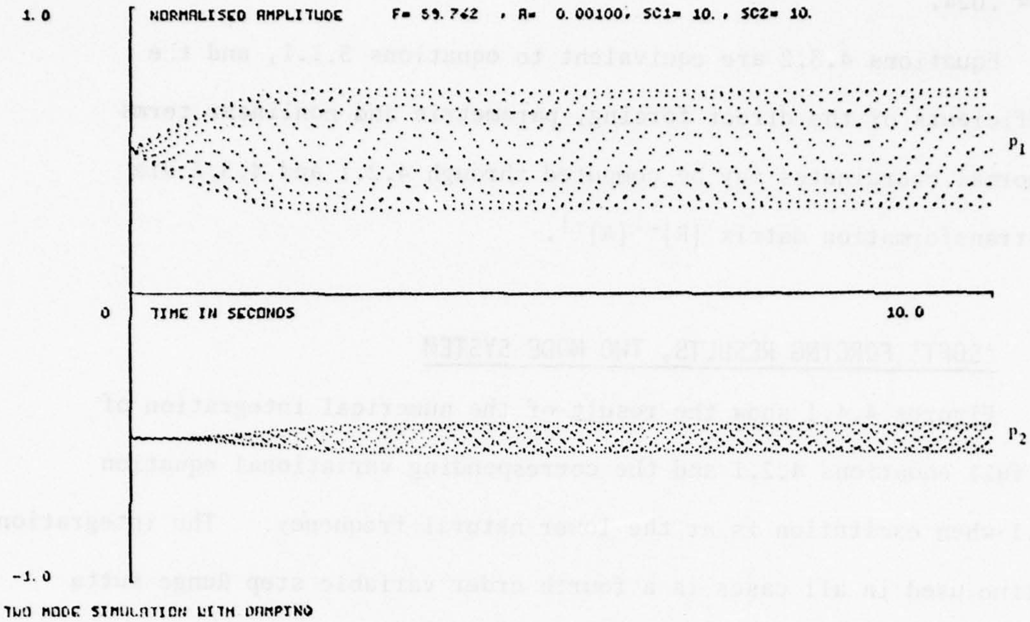
Figures 4.4.1 show the result of the numerical integration of the full equations 4.2.1 and the corresponding variational equation 3.4.1 when excitation is at the lower natural frequency. The integration routine used in all cases is a fourth order variable step Runge Kutta implemented in Fortran on a DEC PDP8 minicomputer.

The points are plotted to the scales shown such that the actual values are given by

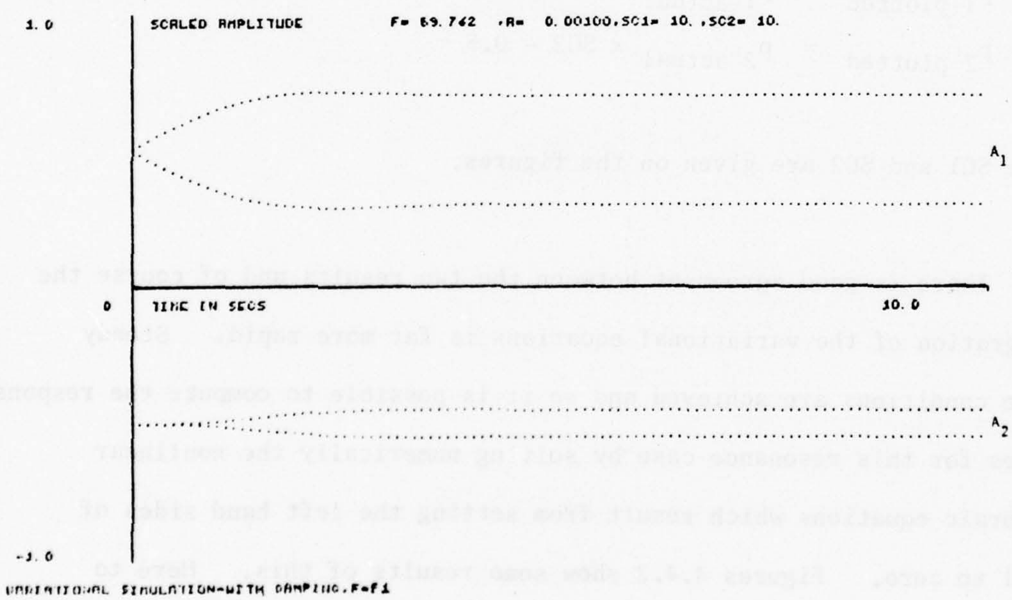
$$p_{1\ \text{plotted}} = p_{1\ \text{actual}} \times SC1 + 0.5$$
$$p_{2\ \text{plotted}} = p_{2\ \text{actual}} \times SC2 - 0.5$$

where SC1 and SC2 are given on the figures.

There is good agreement between the two results and of course the integration of the variational equations is far more rapid. Steady state conditions are achieved and so it is possible to compute the response curves for this resonance case by solving numerically the nonlinear algebraic equations which result from setting the left hand sides of 3.4.1 to zero. Figures 4.4.2 show some results of this. Here to the scales shown the points plotted are A(I) x SC(I). The upper diagram shows the results obtained for the complete system for both cases of excitation increasing and decreasing from the lower natural



Full Equations



'Variational' Equations

Fig. 4.4.1 Excitation at ω_1

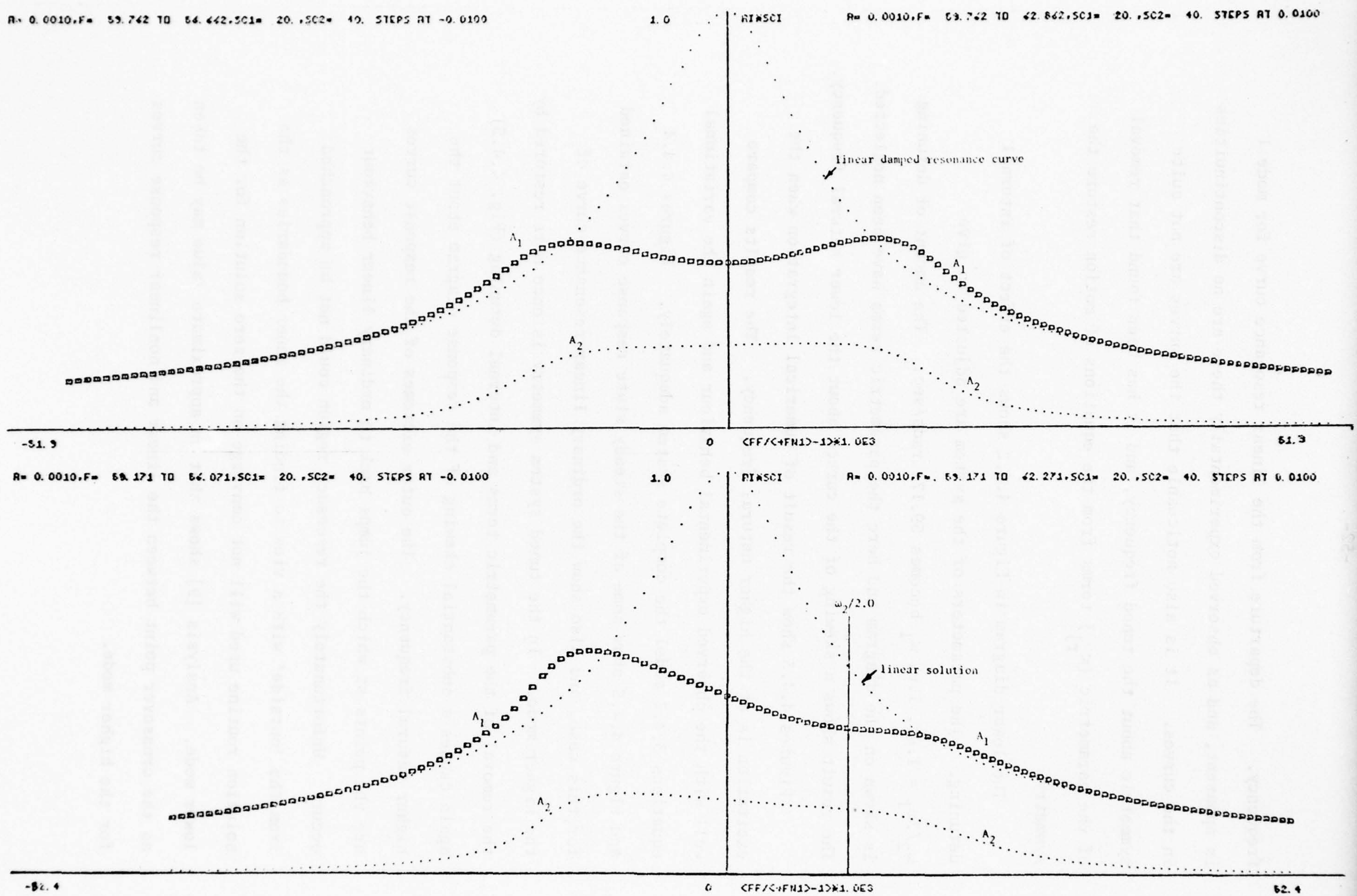
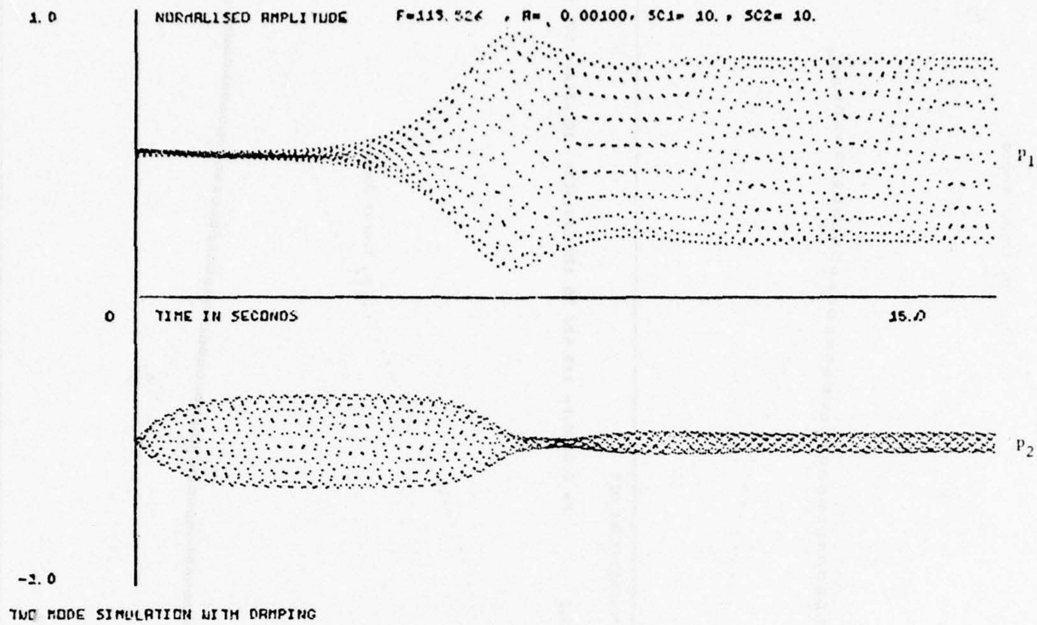


Fig. 4.4.2. Steady state response curves $\Omega_f \sim \omega_1$ - upper, tuned complete system; lower, detuned system. (parametric terms neglected.)

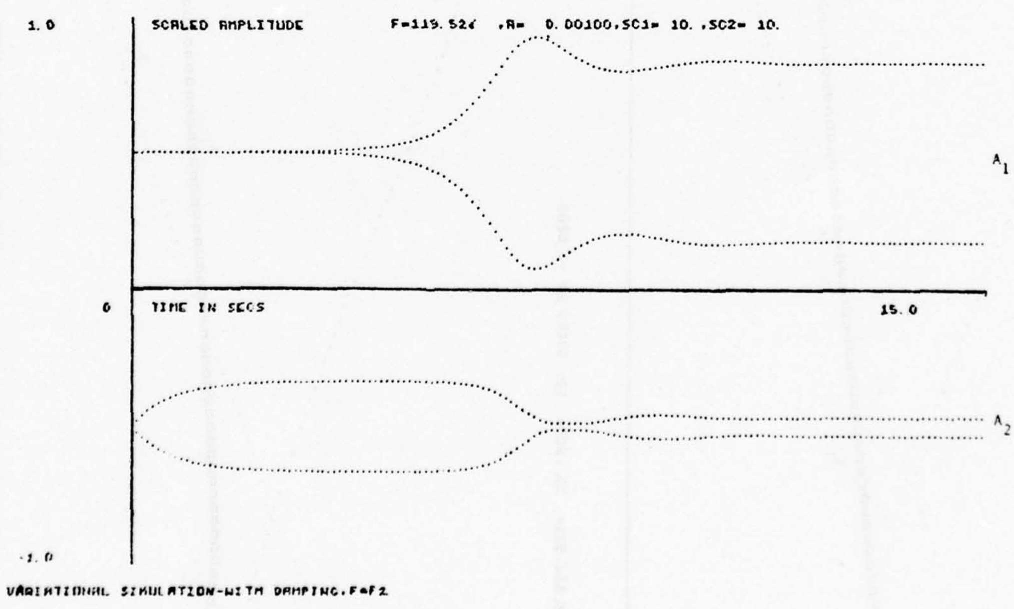
frequency. The departure from the linear resonance curve for mode 1 is apparent, and as observed experimentally there are no discontinuities in the curves. It is also noticeable that the curves are not quite symmetric about the tuned frequency, and it has been found that removal of the parametric (κ_{rj}) terms from the equations of motion restore the symmetry.

The lower diagram in Figure 4.4.2 shows the effect of internal detuning. The parameters of the system are adjusted to give $\omega_2/2\omega_1 = 1.01$, i.e. ω_1 becomes 59.171 rads/sec. The amount of detuning is shown on the diagram and here the parametric terms have been neglected. The result shows a skewing of the curves about the lower natural frequency.

Figures 4.4.3 show the result of numerical integration when the excitation is at the higher natural frequency. The results compare well with the observed experimental behaviour and again the variational equations 3.4.2 model the complete system adequately. Figures 4.4.4 and Figure 4.4.5 show some of the steady state response curves obtained for this case, and also show the ordinary linear resonance curve of the higher mode. In the tuned system symmetry is once more restored by the removal of the parametric terms and internal detuning (Fig. 4.4.5) again causes a substantial skewing of the response diagram about the higher natural frequency. The outer extremes of the response curves are the points at which the jumps back to ordinary linear behaviour occur. Unfortunately the resonance region could not be approached from the 'outside' with a view to finding the inner boundaries as the solution routine used will not converge on the zero solution for the lower mode. Analysis [9] shows that an approximate value may be taken as the crossover point between the linear and nonlinear response curves for the higher mode.



Full Equations



'Variational' Equations

Fig. 4.4.3. Excitation at ω_2.

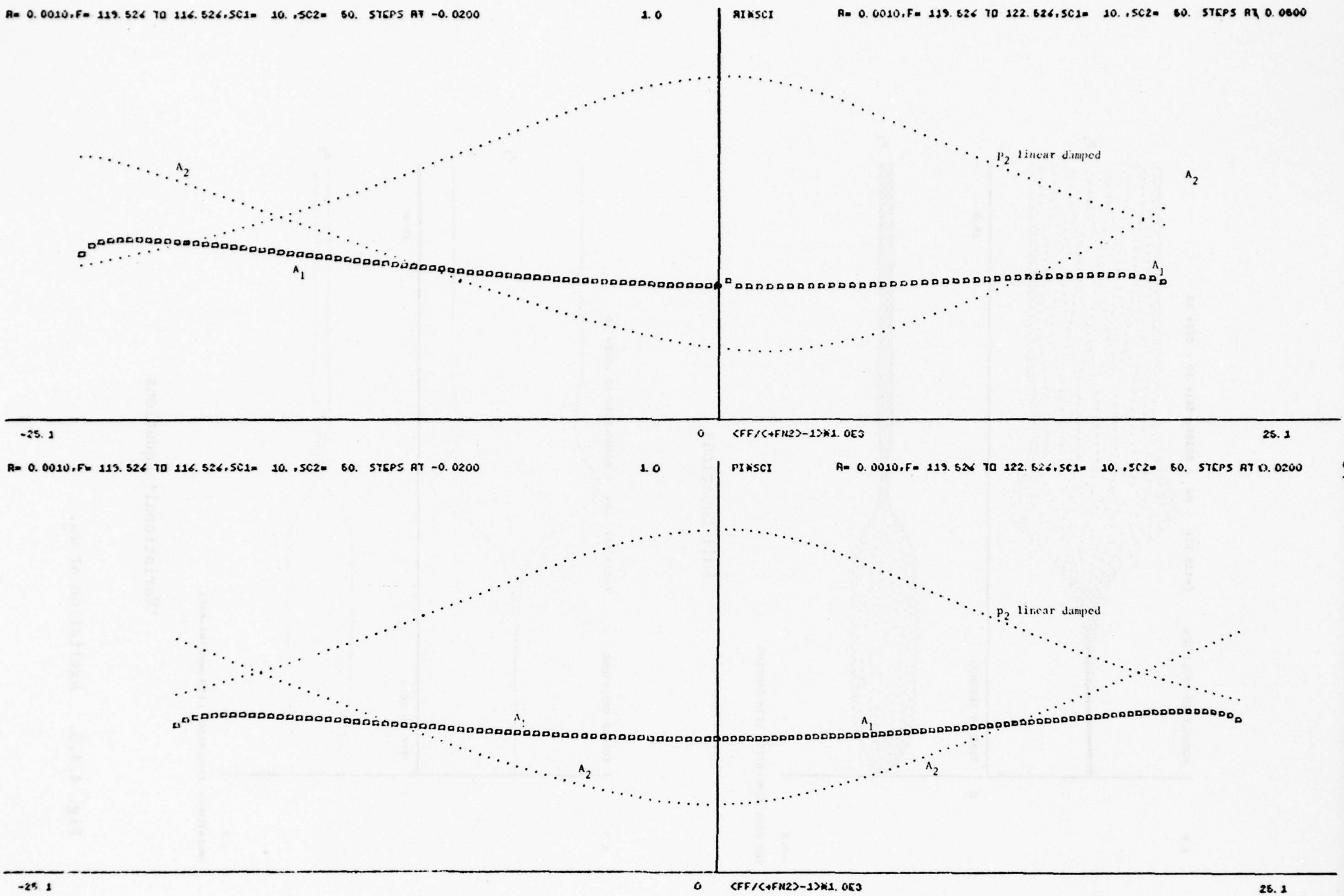


Fig. 4.4.4. Steady state response curves $\Omega_f \sim \omega_1$ - Upper, complete solution; Lower, solution without parametric terms.

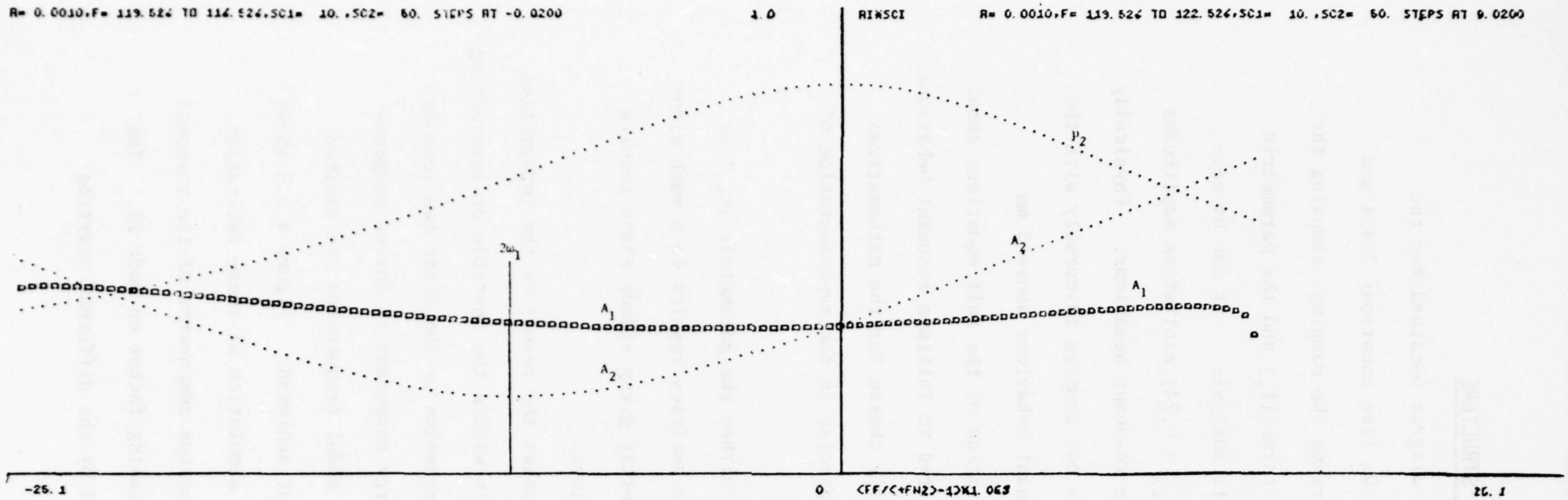


Fig. 4.4.5. Steady State Response curve, detuned system $\Omega_f \sim \omega_2$.

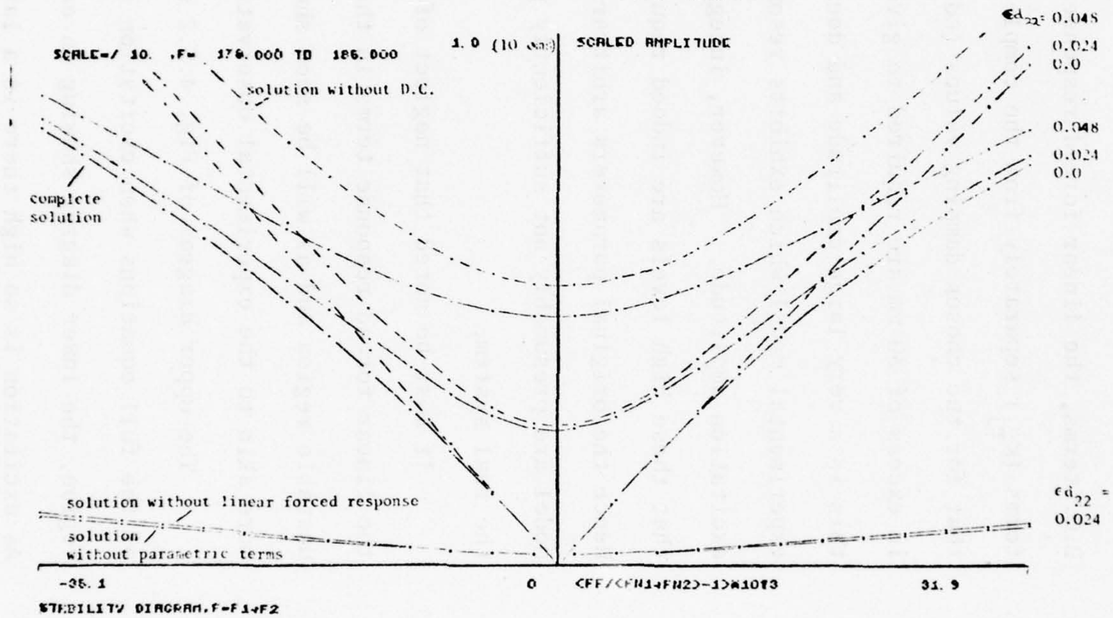


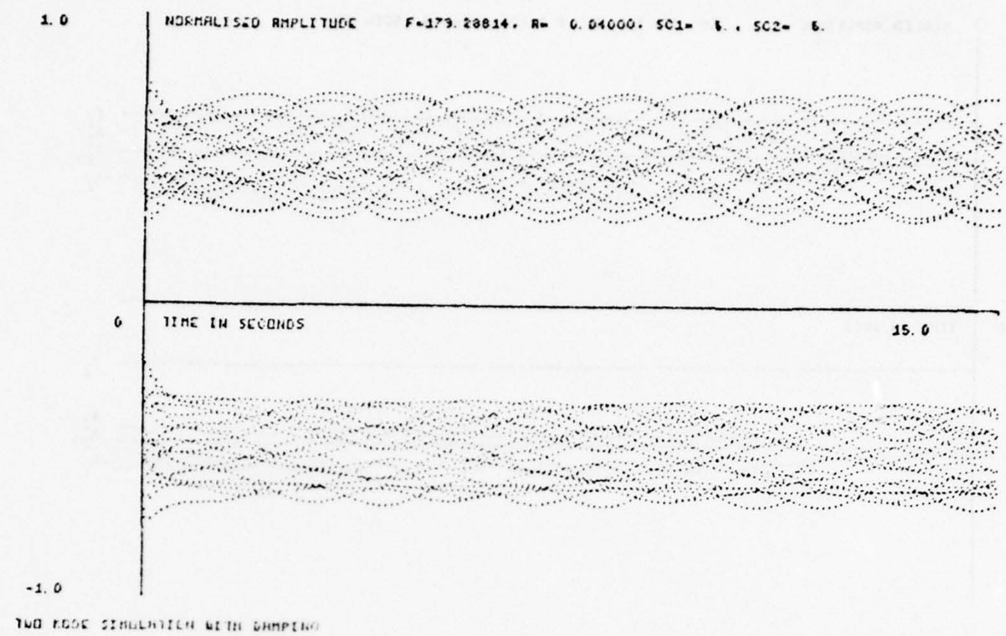
Fig. 4.5.1. Stability Diagram, $\Omega_f = \Omega_1 + \Omega_2$

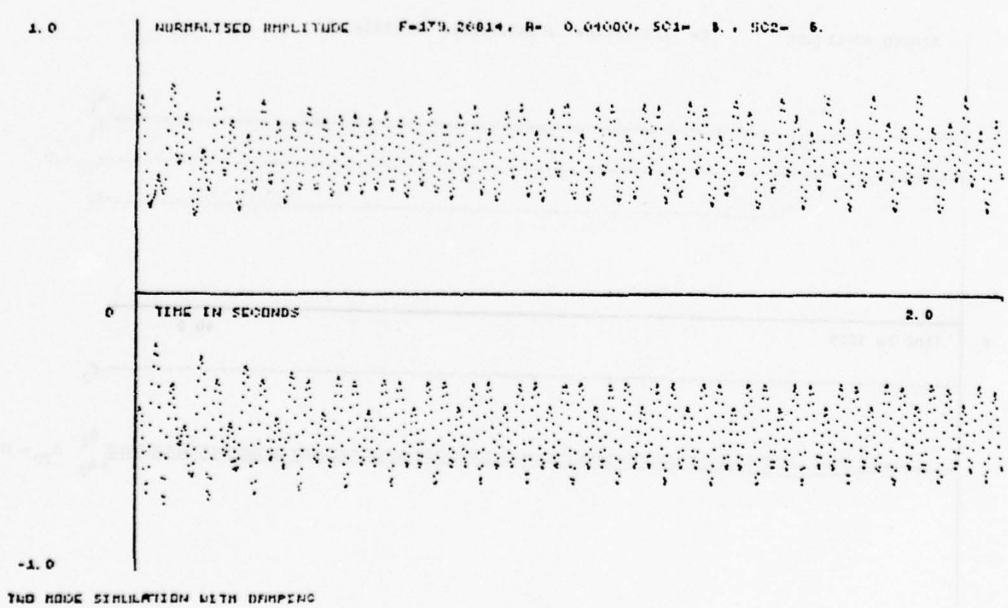
4.5 'HARD' FORCING RESULTS, TWO MODE STRUCTURE

Figure 4.5.1 shows the stability diagram obtained for the combination excitation case, $\Omega_f = \Omega_1 + \Omega_2$ (see equations 3.4.4 and 3.6.11). It shows the effects of varying the damping, removing the D.C. terms, the linear forced response terms (f_r) and the parametric terms (κ_{rj}) separately from the complete analysis. It can be seen that for the chosen damping values ($\epsilon d_{22} = 0.24$) excitation amplitudes in excess of 30 mm are required to give resonant behaviour. Physically this is a very large amplitude and does not compare favourably with the experimental model which exhibits resonant behaviour above 3 mm excitation amplitude. However, integration of the full equations shows that these high levels are indeed required to initiate resonant behaviour. Hence the original parameters arbitrarily chosen for the mathematical model are presumably not sufficiently precise in the representation of the real system.

It must be noted that neglect of either the parametric (κ_{rj}) or the linear forced response terms in the analyses results in a much wider unstable region and as will be seen shortly gives steady state results more akin to the experimental observations.

The upper diagram of Fig. 4.5.2 shows the result of the integration of the full equations when excitation is within the unstable or interacting region, the lower diagram showing an expansion of the first two seconds. As excitation is so high there is a large component of forced response on each mode, but it is clear that the modal frequencies are excited and that steady state conditions are not achieved. Figure 4.5.3 shows the corresponding complete variational simulation at these excitation conditions (equations 3.4.4) and the various components of the response are shown separately (note different scaling factor on mode 2). The different transient behaviour is caused by the different starting





Figs. 4.5.2. Excitation at $\omega_1 + \omega_2$, full equations.

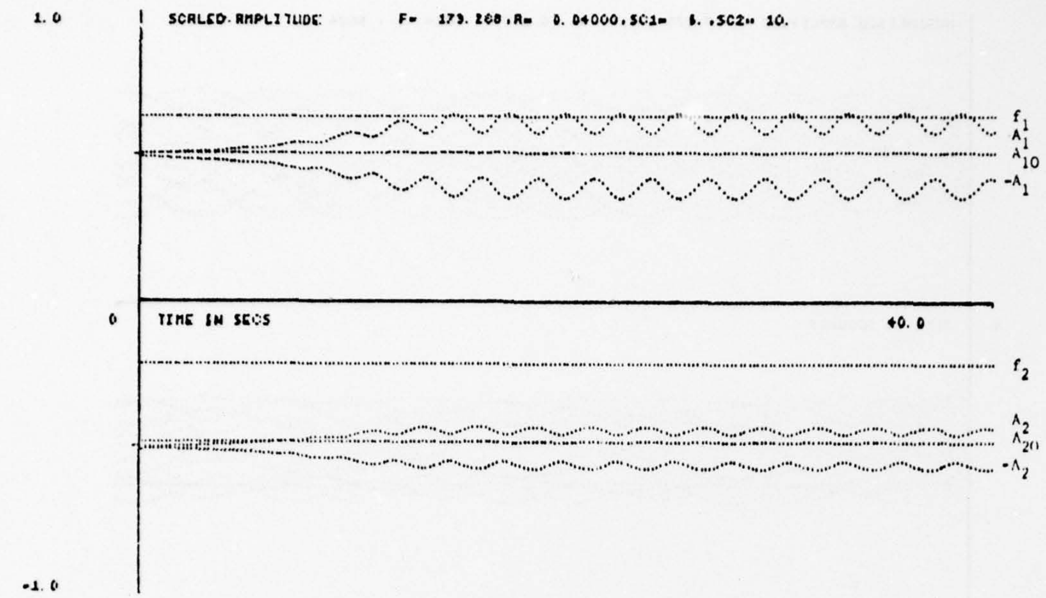


Fig. 4.5.3. Complete variational analysis corresponding to Fig. 4.5.2.

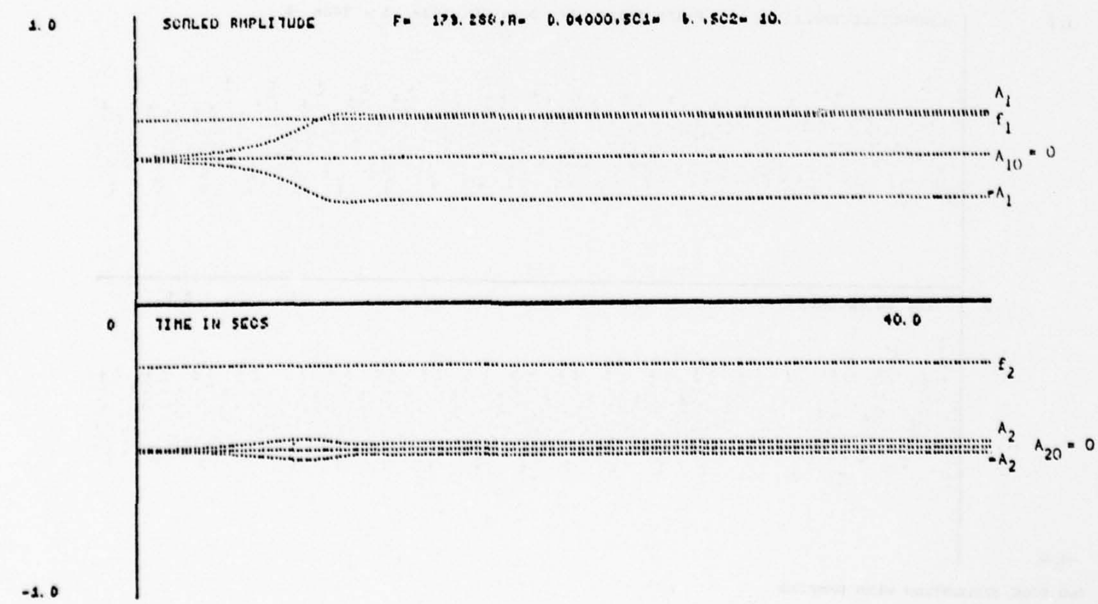


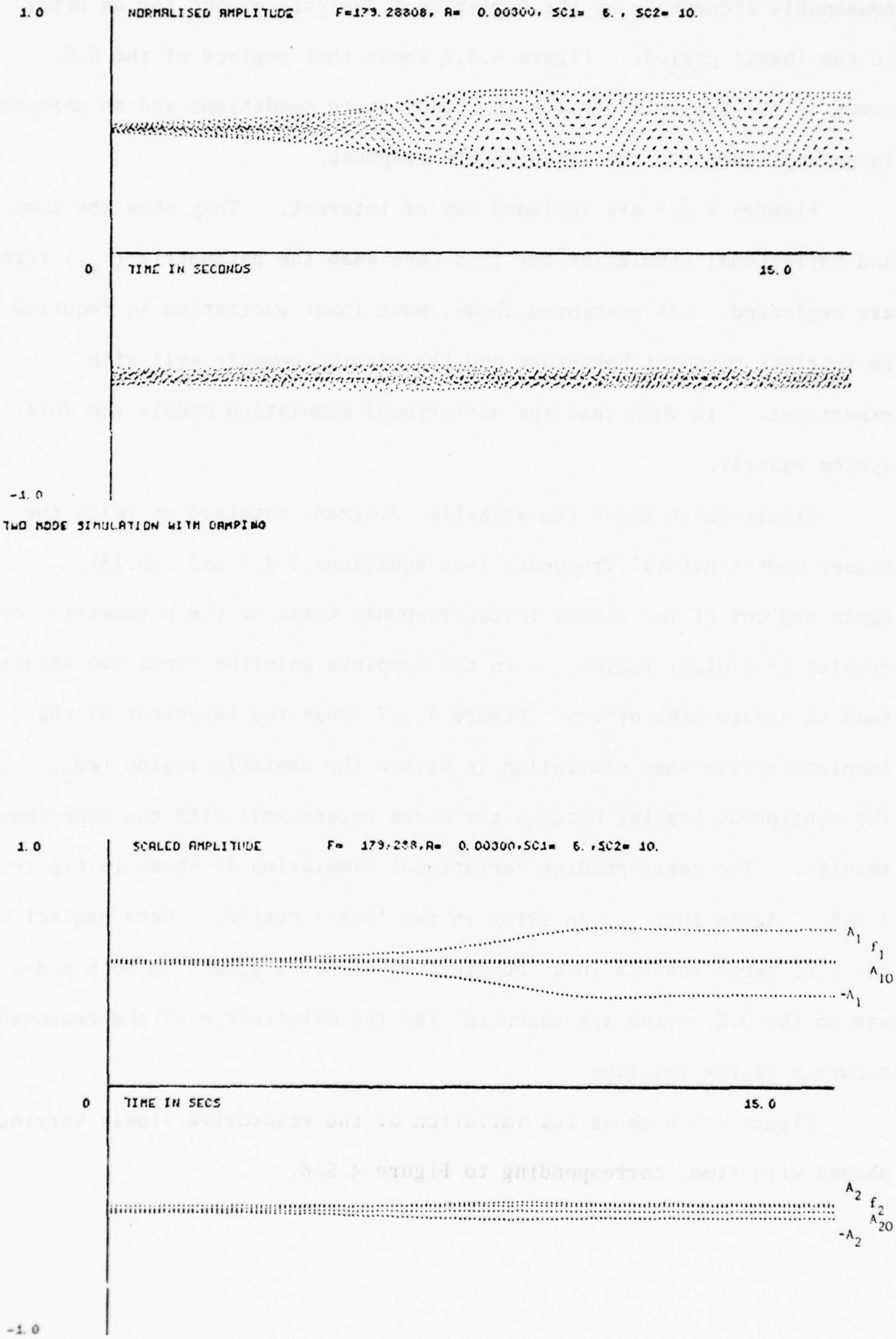
Fig. 4.5.4. Variational analysis with D.C. set to zero

conditions for each solution, but the 'quasi' steady-state is modelled reasonably accurately by the variational analysis except for an error in the 'beat' period. Figure 4.5.4 shows that neglect of the D.C. terms in the analysis results in steady state conditions and an unreasonably large amplitude for the lower mode's response.

Figures 4.5.5 are included out of interest. They show the complete and variational simulation for this case when the parametric (κ_{rj}) terms are neglected. As mentioned above, much lower excitation is required to initiate resonant behaviour and the results compare well with experiment. In this case the variational simulation models the full system exactly.

Figure 4.5.6 shows the stability diagrams obtained at twice the higher mode's natural frequency (see equations 3.4.7 and 3.6.13). Again neglect of the linear forced response terms or the parametric terms results in a wider region, so in the complete solution these two effects tend to negate each other. Figure 4.5.7 shows the behaviour of the complete system when excitation is within the unstable region (ϵd_{22} = .024). The continuous beating between the modes agrees well with the experimental results. The corresponding variational simulation is shown in Figure 4.5.8. Again there is an error in the 'beat' period. Here neglect of the D.C. terms results in a 'beating' exponential growth in both modes, and so the D.C. terms are essential for the maintenance of the reasonable accuracy of the solution.

Figure 4.5.9 shows the variation of the respective slowly varying phases with time, corresponding to Figure 4.5.8.



Figs. 4.5.5. Full and variational equations with parametric (κ_{ij}) terms set to zero, $\Omega_f = \omega_1 + \omega_2$

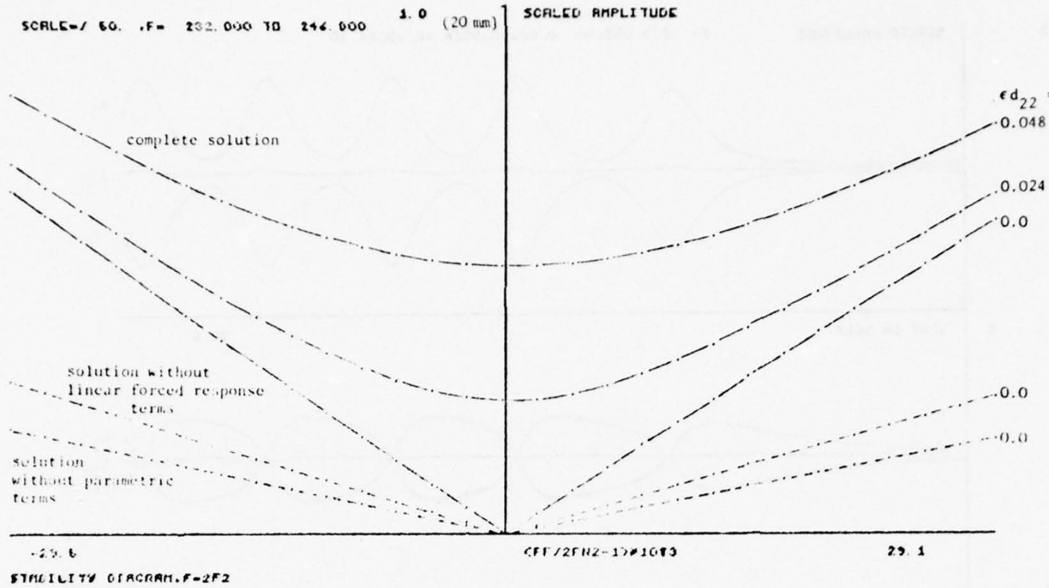


Fig. 4.5.6. Stability Diagram $\Omega_f = 2\Omega_2$

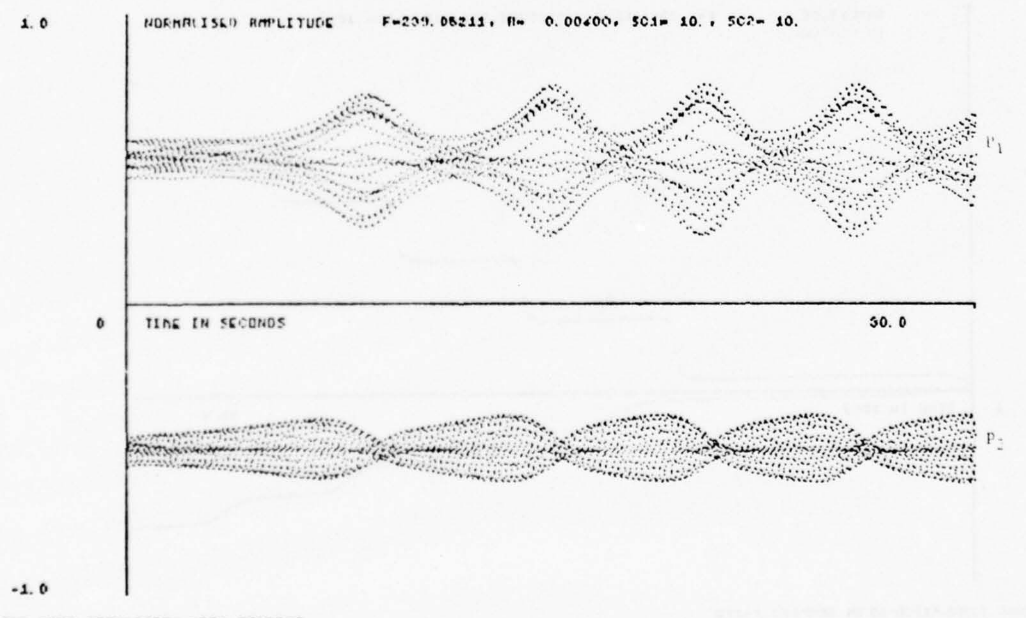


Fig. 4.5.7. Excitation at $2\omega_2$, full equations

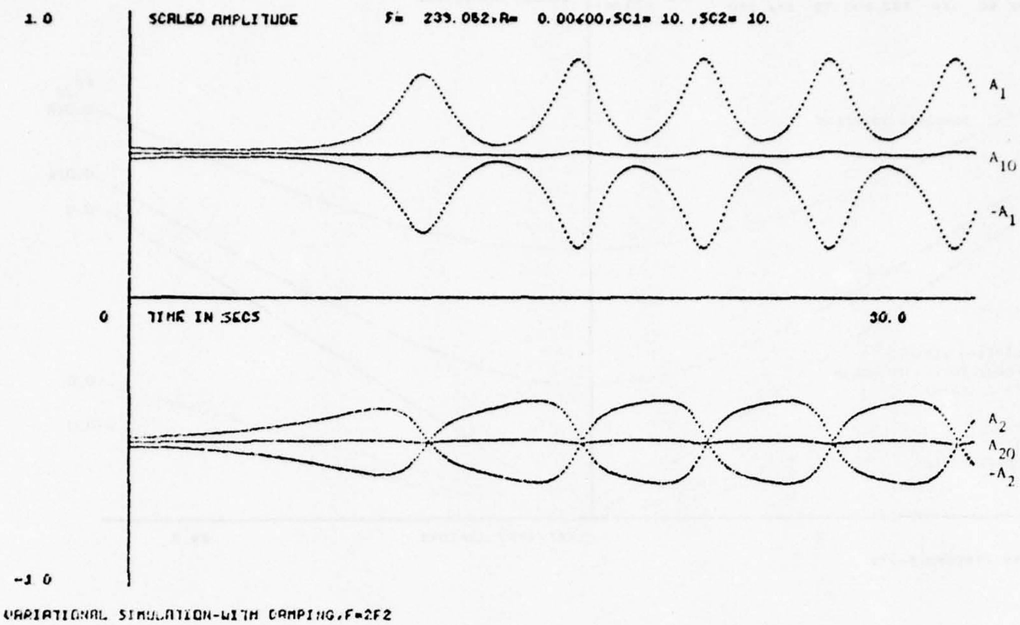


Fig. 4.5.8. Excitation at $2\omega_2$, variational equations

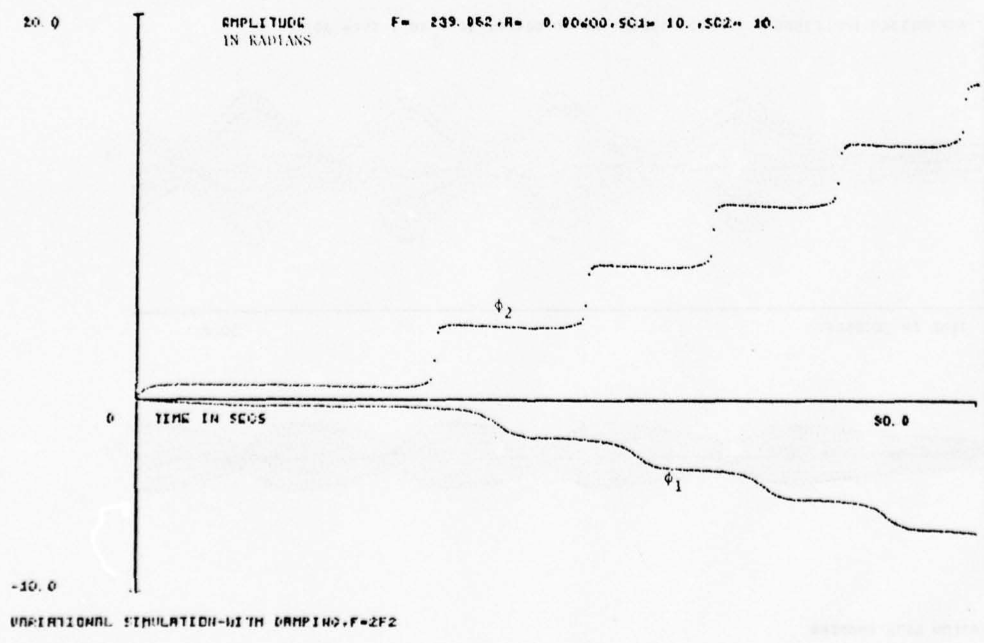


Fig. 4.5.9. Phase plot for Fig. 4.5.8.

4.6 THE FOUR MODE SYSTEM

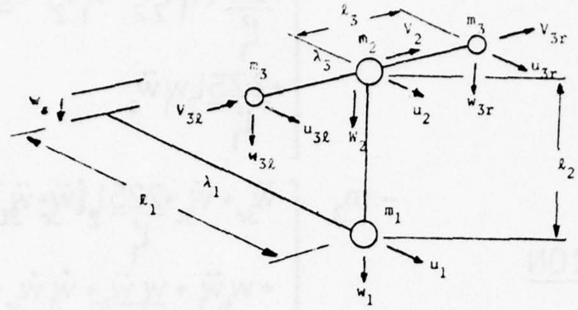


Fig. 4.6.1. Four Mode Theoretical Model

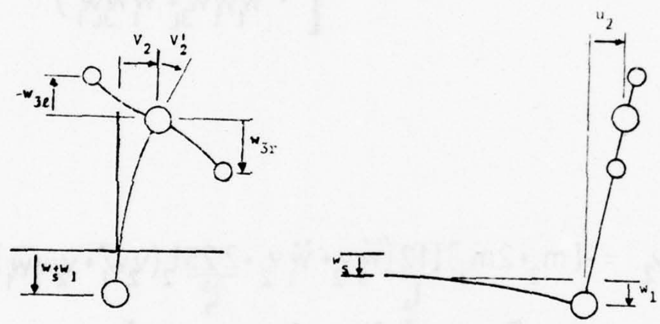


Fig. 4.6.2. Arbitrarily Displaced System

Figures 4.6.1 and 2 show the idealised theoretical model chosen to represent the four mode system. Again the equations of motion are derived inclusive of the quadratic and cubic nonlinearities via formulation of kinetic and potential energy functions and application of Lagrange's equations. Static deflection curves are assumed for each beam. The resulting equations of motion in the four generalised coordinates w_1, v_2, w_{3r}, $w_{3\ell}$ are given below (Equations 4.6.1).

$$[m_1+m_2+2m_3]\cdot\ddot{w}_1+\lambda_1 w_1 = -[m_1+m_2+2m_3]\left[\ddot{w}_s+\frac{1.44}{l_1^2}(w_1^2\ddot{w}_1+w_1\dot{w}_1^2)\right]$$

$$-[m_2+2m_3]\left[\begin{array}{l}\frac{2.25}{l_1^2}l_2^2\ddot{w}_1+\frac{0.9}{l_1^2}l_2\left(w_1\ddot{w}_1+\frac{\dot{w}_1^2}{2}\right)+\frac{1.2}{l_2}(v_2\ddot{v}_2+\dot{v}_2^2)\\ +\frac{2.7}{l_1^2}(w_1v_2\ddot{v}_2+w_1\dot{v}_2^2)+\frac{5.0625}{l_1^4}(w_1^2\ddot{w}_1+w_1\dot{w}_1^2)l_2^2\\ +\frac{2.25}{l_1^2}l_2w_1\ddot{w}_s\end{array}\right]$$

EQUATION OF MOTION

FOR W_1

$$-m_3\left[\begin{array}{l}\ddot{w}_{3r}+\ddot{w}_{3l}+\frac{2.25}{l_1^2}l_2(\ddot{w}_{3r}+\ddot{w}_{3l})w_1-\frac{4.5}{l_1^2}l_2\left(\frac{w_1\ddot{w}_{3l}}{2}+\dot{w}_1\dot{w}_{3l}\right.\\ \left.+w_{3l}\ddot{w}_1+\frac{w_1\ddot{w}_{3r}}{2}+\dot{w}_1\dot{w}_{3r}+w_{3r}\ddot{w}_1\right)\\ +\frac{2.25}{l_1^2}(w_{3r}^2\ddot{w}_1+w_{3r}\dot{w}_1\dot{w}_{3r}+w_{3r}w_1\ddot{w}_{3r}+w_{3l}^2\ddot{w}_1\\ +w_{3l}\dot{w}_1\dot{w}_{3l}+w_{3l}w_1\ddot{w}_{3l})\\ +\frac{3.6}{l_1^2}\left(\frac{w_1^2\ddot{w}_{3r}}{2}+w_1\dot{w}_1\dot{w}_{3r}+w_1w_{3r}\ddot{w}_1+\frac{w_1^2\ddot{w}_{3l}}{2}\right.\\ \left.+w_1\dot{w}_1\dot{w}_{3l}+w_1w_{3l}\ddot{w}_1\right)\end{array}\right]$$

$$[m_2+2m_3]\cdot\ddot{v}_2+\lambda_2v_2=-[m_2+2m_3]\left[\frac{1.2}{l_2}(\ddot{w}_sv_2+\ddot{w}_1v_2)+\frac{2.25}{l_1^2}l_2(v_2\dot{w}_1^2+v_2w_1\ddot{w}_1)+\frac{1.44}{l_2^2}(v_2^2\ddot{v}_2+v_2\dot{v}_2^2)\right]$$

$$-m_3\left[\begin{array}{l}\frac{10.135}{l_2^4}l_3^2(v_2^2\ddot{v}_2+v_2\dot{v}_2^2)+\frac{2.7}{l_2^2}(v_2\dot{w}_{3r}^2+v_2w_{3r}\ddot{w}_{3r}+v_2\dot{w}_{3l}^2+v_2w_{3l}\ddot{w}_{3l})\\ +\frac{1.2}{l_3}(w_{3l}\ddot{w}_{3l}+\dot{w}_{3l}^2-w_{3r}\ddot{w}_{3r}-\dot{w}_{3r}^2)+\frac{1.2}{l_2}v_2(\ddot{w}_{3r}+\ddot{w}_{3l})\end{array}\right]$$

FOR V_2

$$-\lambda_3\left[\frac{4.5}{l_2^2}l_3^2v_2-\frac{1.5}{l_2}l_3(w_{3r}-w_{3l})\right]$$

$$m_3\ddot{w}_{3r}+\lambda_3w_{3r}=-m_3\left[\begin{array}{l}\frac{1.44}{l_3^2}(\ddot{w}_{3r}w_{3r}^2+\dot{w}_{3r}^2w_{3r})-\frac{1.2}{l_3}\left(\ddot{v}_2w_{3r}-\frac{2.25}{l_2^2}l_3(\dot{v}_2^2w_{3r}+v_2\ddot{v}_2w_{3r})\right)\\ +\ddot{w}_s+\ddot{w}_1+\frac{2.25}{l_1^2}l_2(w_1\ddot{w}_1+\dot{w}_1^2)+\frac{1.2}{l_2}(v_2\ddot{v}_2+\dot{v}_2^2)\\ +\frac{2.25}{l_1^2}(w_1^2\ddot{w}_{3r}+w_1\dot{w}_1\dot{w}_{3r}+\ddot{w}_1w_1w_{3r})+\frac{3.6}{l_1^2}(w_1\dot{w}_1^2+w_1^2\ddot{w}_1)\end{array}\right]$$

FOR W_{3r}

$$+\lambda_3\left(\frac{1.5}{l_2}l_3v_2\right)$$

change sign for W_{3l}

4.6.1

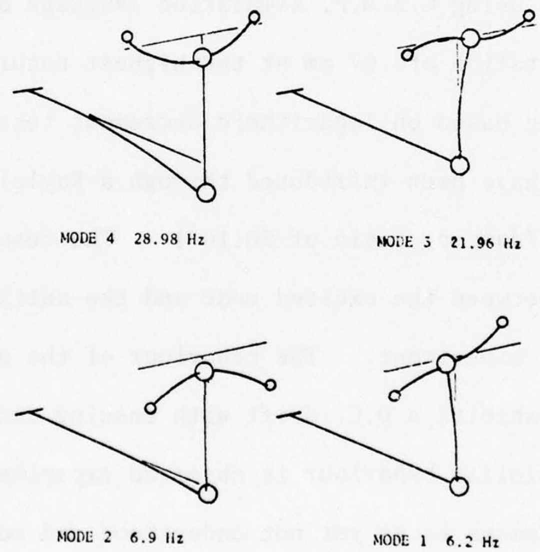
The linear homogeneous equations yield the natural frequencies and normal modes (Section 4.3), and these have been found numerically using standard matrix solution routines. Taking numerical parameters based on experimental measurement of:-

$$m_1 = 144.35 \text{ gms} \qquad \lambda_1 = 1.3686\ 10^7 \text{gm/s} \qquad \ell_1 = 135.5 \text{ mm}$$

$$m_2 = 143.10 \qquad \lambda_2 = 3.67445\ 10^6 \qquad \ell_2 = 66.0$$

$$m_3 = 48.01 \qquad \lambda_3 = 9.28967\ 10^4 \qquad \ell_3 = 97.0,$$

the resulting frequencies and modal matrix is shown in Figure 4.6.3.



Hz	6.2	6.9	21.96	28.98	
w_1	0.0	0.0130	0.0	0.7194	
V_2	-0.0756	0.0	0.9659	0.0	
w_{3r}	-0.7687	0.8999	-0.2403	-0.7639	= [R]
$w_{3\ell}$	0.7687	0.8999	0.2403	-0.7639	

Fig. 4.6.3. Natural Frequencies and Mode Shapes (Computed)

It can be seen that the frequencies obtained are not exactly compatible with those of the experimental model. As expected the higher frequencies are overestimated, and so the desired internal resonance condition ($\omega_1 + \omega_3 \simeq \omega_4$) is not exactly fulfilled. Indeed there is a better internal relationship between modes 2, 3 and 4. However inspection of the nonlinear terms (transformed to normal coordinates) shows that the terms required to give resonant behaviour (e.g. ℓ_{234}, m_{432}, etc) simply do not exist, and so as observed in experiment the response of the system when under excitation at mode 4 is dominated by the interaction with modes 1 and 3, despite the large detuning.

Figure 4.6.4 shows the result of the direct integration of the equations of motion using G.S.M.P. simulation language on an IBM 360/50 computer, with excitation of .07 mm at the highest natural frequency. Here 'modal' damping based on logarithmic decrement tests on the experimental model have been introduced through a Rayleigh Dissipation Function (stiffness/damping ratio of 30.10^3). The result clearly shows interaction between the excited mode and the antisymmetric modes much as observed in experiment. The behaviour of the other mode is interesting as it exhibits a D.C. drift with ensuing small oscillations at about 16 Hz. Similar behaviour is observed experimentally (Figure 2.3.4). This phenomena is as yet not understood and merits further investigation.

Figure 4.6.5 shows the corresponding result obtained from the variational equations (equations 3.4.8). There is good correlation both qualitatively and quantitatively. It was found to be impossible to compute steady state response curves for this case as a true steady state is not achieved. The slowly varying amplitudes in fact exhibit

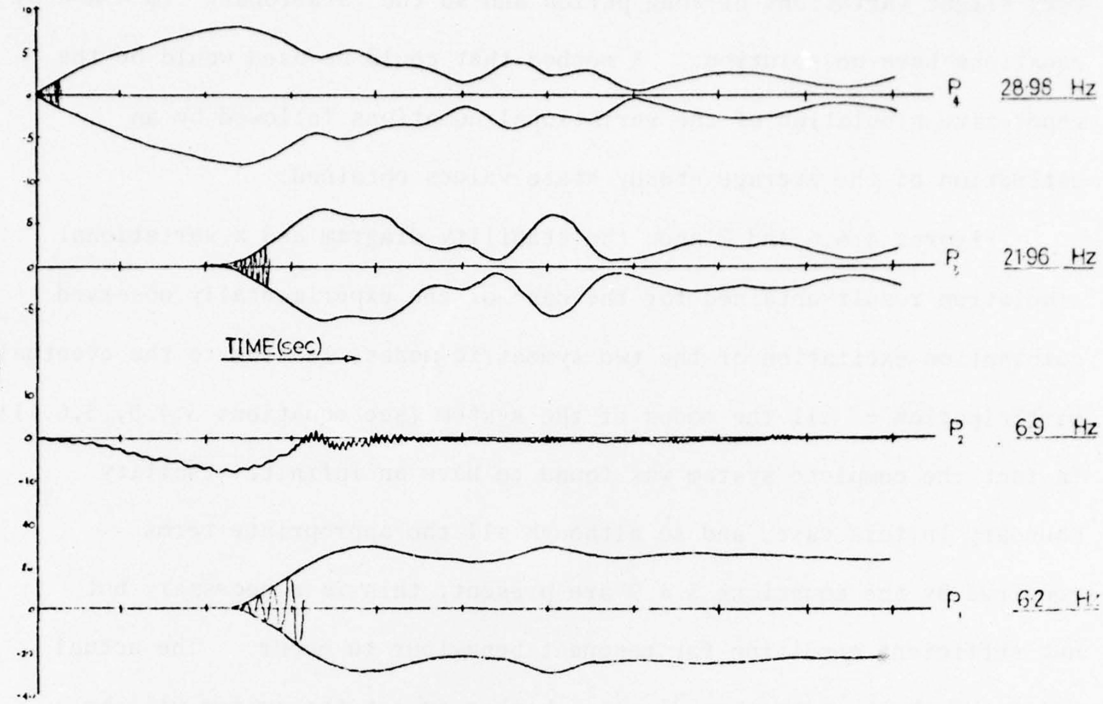


Fig. 4.6.4. CSMP simulation of full equations with quadratic nonlinearities and parametric terms, excitation at 28.98 Hz, 0.07 mm.

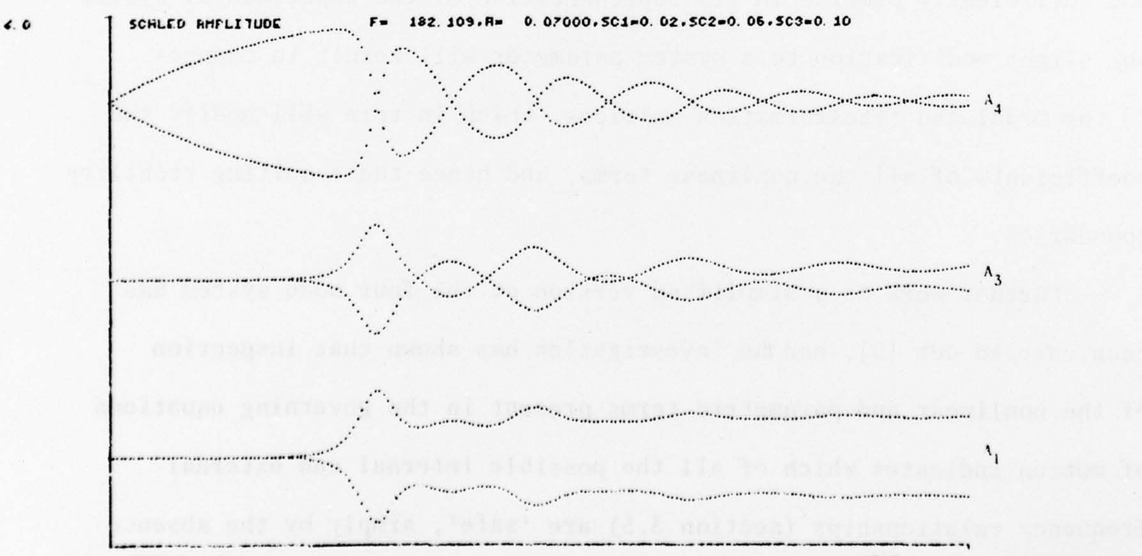


Fig. 4.6.5. Variational simulation corresponding to Fig. 4.6.4.

very slight variations of long period and so the 'stationary' ($\dot{\phi} = \dot{A} = 0$) equations have no solution. A method that could be used would be the repetetive simulation of the variational equations followed by an estimation of the average steady state values obtained.

Figures 4.6.6 and 7 show the stability diagram and a variational simulation result obtained for the case of the experimentally observed combination excitation of the two symmetric modes, leading to the eventual participation of all the modes of the system (see equations 3.4.9, 3.6.11). In fact the complete system was found to have an infinite stability boundary in this case, and so although all the appropriate terms required by the equations 3.4.9 are present, this is a necessary but not sufficient condition for resonant behaviour to occur. The actual values of these parameters dictate whether or not the system will be unstable. The figures shown were obtained, purely for the purposes of example, by neglect of the linear forced response terms in the analysis, i.e. the system is reduced to one having simple 'parametric' (κ_{rj}) terms only. Hence once again the theoretical model is presumably not sufficiently precise in its representation of the experimental system. Any slight modification to a system parameter will result in changes to the modal and transformation matrices, which in turn will modify the coefficients of all the nonlinear terms, and hence the resulting stability boundaries.

Further work on a simplified version of the four mode system has been carried out [9], and the investigation has shown that inspection of the nonlinear and parametric terms present in the governing equations of motion indicates which of all the possible internal and external frequency relationships (section 3.5) are 'safe', simply by the absence of the appropriate terms in the relevant variational equations. Hence the frequency relationships which should be avoided by the designer

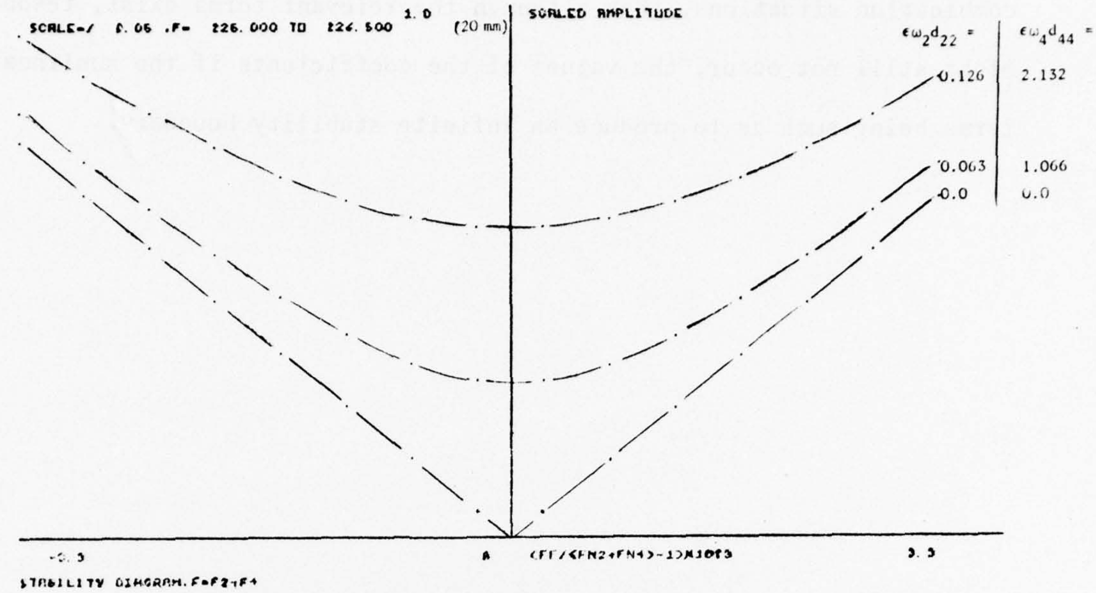


Fig. 4.6.6. Stability Diagram, Excitation Ω_f near $\omega_2 + \omega_4$ (f_2 and f_4 taken arbitrarily as zero to give instability)

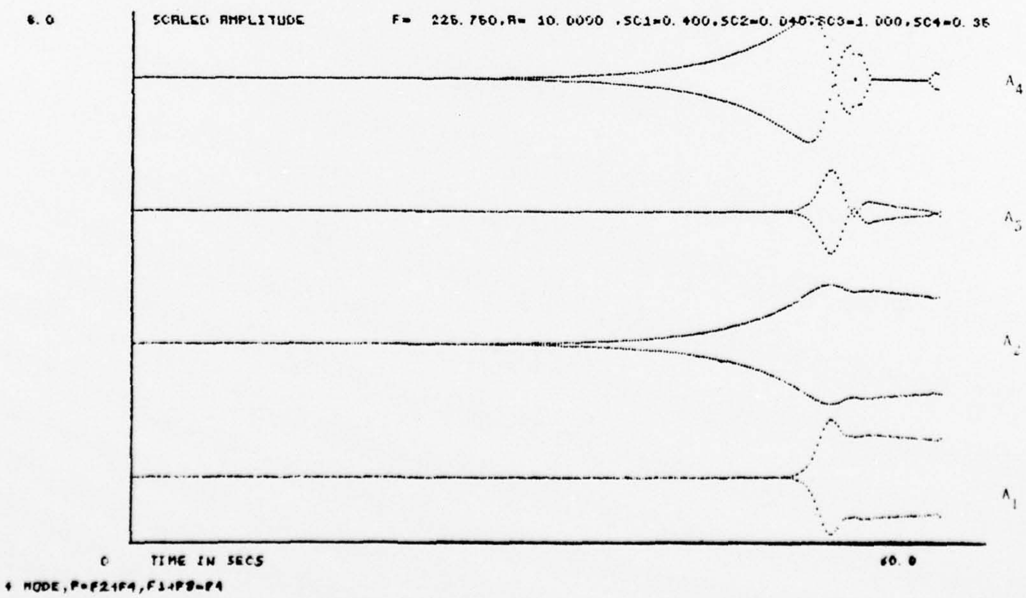


Fig. 4.6.7. Variational simulation when within unstable region given by Fig. 4.6.6. ($\omega_2 d_{22}$ = .063, $\omega_1 d_{11}$ = 1.066)

can be deduced. However, again it has been found, particularly in the combination situations, that although the relevant terms exist, resonance might still not occur, the values of the coefficients if the nonlinear terms being such as to produce an infinite stability boundary.

5. CONCLUSIONS

Parametric and nonlinear resonance behaviour in structures has been illustrated experimentally, and it has been shown that quadratic inertial nonlinearities are of primary significance in determining the response. The resonance conditions and variational equations listed in section 3 (and Appendix I) can be adapted and applied to any multimode structural system having quadratic inertial nonlinearities and linear parametric terms subjected to harmonic excitation. The integrity of the solution technique has been demonstrated by its application to the two and four mode theoretical models. The important factors in maintaining the accuracy in the 'hard' forcing cases are the inclusion of the D.C. terms and the forced response components, the latter being assumed in this first approximation to be fixed at the linear values. The inclusion of the forced response terms in the generating solution give rise to terms of nonlinear origin which accumulate with the ordinary linear parametric excitation terms in the variational equations. The combination of these two separate effects may either reduce or increase the resonance region for a given structure depending upon the values of certain of the nonlinear coupling terms and the forced response of <u>all</u> the modes of the structure (see $P_{i,j}$, equation 3.6.1).

In the 'soft' forcing cases there is good correlation between the theoretical and experimental results for both the two and four mode systems. However, in some of the 'hard' forcing cases (notably the combination case), although there is good correlation between the full equations of motion and the corresponding variational equations,

comparison with the experimental results is relatively poor. This has been attributed to insufficiently precise representation of the experimental systems. However in these cases, particularly in those under intense excitation, it is possible that the cubic nonlinearities may also be important in determining the response, as the resonance conditions that are important for the quadratics are also important for cubics by virtue of the internal resonance conditions (e.g. $\Omega_f = \Omega_1 + \Omega_2 = 3\Omega_1$, as $\Omega_2 = 2\Omega_1$). The cubic nonlinearities can easily be incorporated in the analysis, but this will naturally lead to much more complex sets of variational equations. The equations of motion of most space structures will involve the quadratic nonlinearities studied here and in general they will be the most important nonlinear influence on the response. Higher order nonlinearities will however also be present and under certain circumstances may require to be taken into account. Cubic nonlinearities will introduce further possible external and internal resonance conditions each involving up to three and four modes respectively. Consideration of even higher order nonlinearities will introduce more resonance conditions of increasing complexity.

The experimental work indicated the form of the relationship between the excitation and response frequencies in multimode systems. The resonance conditions between the excitation frequency and the natural frequencies (external) or amongst the natural frequencies themselves (internal) need not be met precisely. It is sufficient that they should be met approximately, the response frequencies are then found to be related to the exciting frequency in such a way that they form a linear relationship with exact integer coefficients. This result is utilised in the solution procedure.

The possibility of the occurrence of 'cascading' type resonant behaviour has also been shown. It may be concluded that for more complicated structures having a multiplicity of modes and particularly under intense excitation, this behaviour may be exhibited often. Indeed, preliminary experimental investigations into thin-walled box type structures under intense 'parametric' excitation have shown that resonance in and interaction between the coupled bending and cross-section distortion modes is quite common, and in general the resulting motion involves several modes and is non-stationary. This, along with the combined direct and parametric excitation of such structures is a promising area for further work. The other area of immediate interest for further research lies in the generalisation of the excitation from the single harmonic considered here to multifrequency excitation, particularly when the forced and parametric excitations have different periods. A single mode system under such excitation is considered in [10].

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APPENDIX I

VARIATIONAL EQUATIONS FOR THE THREE MODE SYSTEMS (Section 3.4.2)

I.1 $\Omega_f = \Omega_1,\ \Omega_1 + \Omega_2 = \Omega_3$ (soft forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 + F_1\cos\phi_1 - \frac{\epsilon}{2}[N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_1\Omega_1 = F_1\sin\phi_1 - \frac{\epsilon}{2}[-N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2\dot{A}_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_3\cos(\phi_3 - \phi_2) + N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[-P_2A_3\sin(\phi_3 - \phi_2) - N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[P_3A_2\cos(\phi_3 - \phi_2) + N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[P_3A_2\sin(\phi_3 - \phi_2) + N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3 d_{33}A_3]$$

where $P_2 = \kappa_{23}$

$P_3 = \kappa_{32}$

$N_1 = (m_{123} + m_{132})\Omega_2\Omega_3 - \ell_{123}\Omega_3^2 - \ell_{132}\Omega_2^2$

$N_2 = (m_{213} + m_{231})\Omega_1\Omega_3 - \ell_{231}\Omega_1^2 - \ell_{213}\Omega_3^2$

$N_3 = -(m_{312} + m_{321})\Omega_1\Omega_2 - \ell_{312}\Omega_2^2 - \ell_{321}\Omega_1^2$

I.2 $\Omega_f = \Omega_2,\ \Omega_1 + \Omega_2 = \Omega_3$ (soft forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_3\cos(\phi_3 - \phi_1) + N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-P_1A_3\sin(\phi_3 - \phi_1) - N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 + F_2\cos\phi_2 - \frac{\epsilon}{2}[N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_2\Omega_2 = F_2\sin\phi_2 - \frac{\epsilon}{2}[-N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[P_3A_1\cos(\phi_3 - \phi_1) + N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)]$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[P_3A_1\sin(\phi_3 - \phi_1) + N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3d_{33}A_3]$$

where $P_1 = \kappa_{13}$, $P_2 = \kappa_{31}$ and $N_{1,2,3}$ as in I.1.

I.3 $\Omega_f = 2\Omega_1$, $\Omega_1 + \Omega_2 = \Omega_3$ (hard forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_1\cos2\phi_1 + N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[P_1A_1\sin2\phi_1 - N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[-N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3d_{33}A_3]$$

where $N_{1,2,3}$ are as in I.1 and

$$P_1 = \kappa_{11} - \ell_{111}f_1(\Omega_1^2 + \Omega_f^2) - \ell_{112}f_2\Omega_f^2 - \ell_{121}f_2\Omega_1^2 - \ell_{113}f_3\Omega_f^2$$

$$-\ell_{131}f_3\Omega_1^2 + 2m_{111}f_1\Omega_1\Omega_f + (m_{112} + m_{121})f_2\Omega_1\Omega_f + (m_{113} + m_{131})f_3\Omega_1\Omega_f$$

I.4 $\Omega_f = 2\Omega_2,\ \Omega_1 + \Omega_2 = \Omega_3$ ('hard' forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_2\cos 2\phi_2 + N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_2\sin 2\phi_2 - N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3d_{33}A_3]$$

where $P_2 = \kappa_{22} - \ell_{222}f_2(\Omega_2^2 + \Omega_f^2) - f_1(\ell_{212}\Omega_2^2 + \ell_{221}\Omega_f^2) - f_3(\ell_{232}\Omega_2^2 +$

$+ \ell_{223}\Omega_f^2) + 2m_{222}f_2\Omega_2\Omega_f + (m_{212} + m_{221})f_1\Omega_2\Omega_f + (m_{223} +$

$+ m_{232})f_3\Omega_2\Omega_f$

and $N_{1,2,3}$ are as in I.1

I.5 $\Omega_f = 2\Omega_3,\ \Omega_1 + \Omega_2 = \Omega_3$ (hard forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[-N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[P_3A_3\cos 2\phi_3 + N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[P_3A_3\sin 2\phi_3 + N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3 d_{33}A_3]$$

where $N_{1,2,3}$ are as in I.1, and

$$P_3 = \kappa_{33} - \ell_{333}f_3(\Omega_3^2 + \Omega_f^2) - \ell_{331}f_1\Omega_f^2 - \ell_{313}f_1\Omega_3^2 - \ell_{332}f_2\Omega_f^2$$

$$-\ell_{323}f_2\Omega_3^2 + 2m_{333}f_3\Omega_3\Omega_f + (m_{313} + m_{331})f_1\Omega_3\Omega_f + (m_{323} + m_{332})f_2\Omega_3\Omega_f$$

<u>I.6 $\quad \Omega_f = \Omega_2 - \Omega_1, \quad \Omega_1 + \Omega_2 = \Omega_3$</u> (hard forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[P_1A_2\cos(\phi_2 - \phi_1) + N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-P_1A_2\sin(\phi_2 - \phi_1) - N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_1\cos(\phi_2 - \phi_1) + N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_1\sin(\phi_2 - \phi_1) - N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3 d_{33}A_3]$$

where $N_{1,2,3}$ are as in I.1, and

$$P_1 = \kappa_{12} - \ell_{122}f_2(\Omega_2^2 + \Omega_f^2) - \ell_{121}f_1\Omega_f^2 - \ell_{112}f_1\Omega_2^2 - \ell_{123}f_3\Omega_f^2 -$$

$$-\ell_{132}f_3\Omega_2^2 + 2m_{122}f_2\Omega_2\Omega_f + (m_{121} + m_{112})f_1\Omega_2\Omega_f + (m_{123} + m_{132})f_3\Omega_2\Omega_f$$

$$P_2 = \kappa_{21} - \ell_{211}f_1(\Omega_1^2 + \Omega_f^2) - \ell_{212}f_2\Omega_f^2 - \ell_{221}f_2\Omega_1^2 - \ell_{231}f_3\Omega_1^2 -$$

$$-\ell_{213}f_3\Omega_f^2 - 2m_{211}f_1\Omega_1\Omega_f - (m_{212} + m_{221})f_2\Omega_1\Omega_f - (m_{231} + m_{213})f_3\Omega_1\Omega_f.$$

<u>I.7</u> $\underline{\Omega_f = \Omega_3 + \Omega_2, \quad \Omega_1 + \Omega_2 = \Omega_3}$ (hard forcing)

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}[N_1A_2A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}[-N_1A_2A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}[P_2A_3\cos(\phi_2 + \phi_3) + N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}[P_2A_3\sin(\phi_2 + \phi_3) - N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}[P_3A_2\cos(\phi_2 + \phi_3) + N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}[P_3A_2\sin(\phi_2 + \phi_3) + N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3d_{33}A_3]$$

where $N_{1,2,3}$ are as in I.1

and $$P_2 = \kappa_{23} - \ell_{213}f_1\Omega_3^2 - \ell_{231}f_1\Omega_f^2 - \ell_{233}f_3(\Omega_3^2 + \Omega_f^2) - \ell_{232}f_2\Omega_f^2 -$$

$$-\ell_{223}f_2\Omega_3^2 + (m_{232} + m_{223})f_2\Omega_f\Omega_3 + (m_{213} + m_{231})f_1\Omega_3\Omega_f + 2m_{233}f_3\Omega_3\Omega_f$$

$$P_3 = \kappa_{32} - \ell_{312}f_1\Omega_2^2 - \ell_{321}f_1\Omega_f^2 - \ell_{322}f_2(\Omega_2^2 + \Omega_f^2) - \ell_{332}f_3\Omega_2^2 -$$

$$-\ell_{323}f_3\Omega_f^2 + (m_{332} + m_{323})f_3\Omega_2\Omega_f + (m_{312} + m_{321})f_1\Omega_2\Omega_f + 2m_{322}f_2\Omega_2\Omega_f.$$

$\underline{\text{I.8}}$ $\underline{\Omega_f = \Omega_3 + \Omega_1, \quad \Omega_1 + \Omega_2 = \Omega_3}$

$$-2A_1\Omega_1\dot{\phi}_1 = (\Omega_1^2 - \omega_1^2)A_1 - \frac{\epsilon}{2}\,[P_1A_3\cos(\phi_1 + \phi_3) + N_1A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_1$$

$$-2\dot{A}_1\Omega_1 = -\frac{\epsilon}{2}\,[P_1A_3\sin(\phi_1 + \phi_3) - N_1A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_1\omega_1 d_{11}A_1]$$

$$-2A_2\Omega_2\dot{\phi}_2 = (\Omega_2^2 - \omega_2^2)A_2 - \frac{\epsilon}{2}\,[N_2A_1A_3\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_2$$

$$-2\dot{A}_2\Omega_2 = -\frac{\epsilon}{2}\,[-N_2A_1A_3\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_2\omega_2 d_{22}A_2]$$

$$-2A_3\Omega_3\dot{\phi}_3 = (\Omega_3^2 - \omega_3^2)A_3 - \frac{\epsilon}{2}\,[P_3A_1\cos(\phi_1 + \phi_3) + N_3A_1A_2\cos(\phi_3 - \phi_2 - \phi_1)] + \nabla_3$$

$$-2\dot{A}_3\Omega_3 = -\frac{\epsilon}{2}\,[P_3A_1\sin(\phi_1 + \phi_3) + N_3A_1A_2\sin(\phi_3 - \phi_2 - \phi_1) - 2\Omega_3\omega_3 d_{33}A_3]$$

where $N_{1,2,3}$ are as in I.1

and $P_1 = \kappa_{13} - \ell_{123}f_2\Omega_3^2 - \ell_{132}f_2\Omega_f^2 - \ell_{133}f_3(\Omega_3^2 + \Omega_f^2) - \ell_{113}f_1\Omega_3^2 -$

$-\ell_{131}f_1\Omega_f^2 + (m_{113} + m_{131})f_1\Omega_f\Omega_3 + (m_{123} + m_{132})f_2\Omega_f\Omega_3 + 2m_{133}f_3\Omega_3\Omega_f,$

$P_3 = \kappa_{31} - \ell_{312}f_2\Omega_f^2 - \ell_{321}f_2\Omega_1^2 - \ell_{311}f_1(\Omega_1^2 + \Omega_f^2) - \ell_{313}f_3\Omega_f^2$

$-\ell_{331}f_3\Omega_1^2 + (m_{331} + m_{313})f_3\Omega_1\Omega_f + (m_{312} + m_{321})f_2\Omega_1\Omega_f + 2m_{311}f_1\Omega_1\Omega_f.$